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A GENERALIZATION OF PARALLELISM IN RIEMANNIAN GEOMETRY, THE C^{ω} CASE

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Abstract. The concept of parallelism along a curve in a Riemannian manifold is generalized to parallelism along higher dimensional immersed submanifolds in such a way that the minimal immersions are self parallel and hence correspond to geodesics. Let $g: N \to M$ be a (not necessarily isometric) immersion of Riemannian manifolds. Let $G: T(N) \to T(M)$ be a tangent bundle isometry along g, that is, G covers g and maps fibers isometrically. By mimicing the construction used for isometric immersions, it is possible to define the mean curvature vector field of G. G is said to be parallel along g if this vector field vanishes identically. In particular, minimal immersions have parallel tangent maps. For curves, it is shown that the present definition reduces to the definition of Levi-Civita. The major effort is directed toward generalizations, in the real analytic case, of the two basic theorems for parallelism. On the one hand, the existence and uniqueness theorem for a geodesic in terms of data at a point extends to the well-known existence and uniqueness of a minimal immersion in terms of data along a codimension one submanifold. On the other hand, the existence and uniqueness theorem for a parallel unit vector field along a curve in terms of data at a point extends to a local existence and uniqueness theorem for a parallel tangent bundle isometry in terms of mixed initial and partial data. Since both extensions depend on the Cartan-Kahler Theorem, a procedure is developed to handle both proofs in a uniform manner using fiber bundle techniques.

0. **Introduction.** The geodesics in a Riemannian manifold M, of dimension m, can be described in two independent ways: they are the critical points for the variational problem for minimal 1-dimensional area with fixed endpoints and they are also the auto-parallels, the curves whose tangent vector fields are parallel. More generally, the critical points for the variational problem for minimal p dimensional area, $1 \leq p < m$, with fixed boundary are the minimal immersions.

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It is our intention to generalize the notion of parallelism so that in each dimension the minimal immersions are also the auto-parallels.

Briefly, this generalization goes as follows. Given an (not necessarily isometric) immersion $g: N^p \to M^m$ of Riemannian manifolds, a tangent bundle isometry (TBI) along g is a map between tangent bundles, $G: T(N) \to T(M)$, that covers g and maps fibers isometrically. If g is an isometric immersion the tangent map $Tg: T(N) \to T(M)$ is a TBI along g. The mean curvature vector field of a TBI is defined in a way that generalizes the mean curvature field of an isometric immersion. G is parallel if the mean curvature vanishes everywhere; see §3.1. In particular, minimal immersions have parallel tangent maps. It is shown in §3.2 that the definition generalizes the usual notion of a parallel unit vector field along an immersed curve.

The main body of this paper is concerned with generalizations, in the C^ω case, of the two existence and uniqueness theorems related to parallelism.

The first theorem asserts the existence and uniqueness of a geodesic in terms of initial data: its tangent vector at a single point. The generalization, Theorem A (p), is well known: A p-dimensional real analytic minimal submanifold is (locally) uniquely determined by initial data, namely its tangent plane along N^{p-1}, a codimension 1 submanifold. See §4.3.

The second theorem asserts the existence and uniqueness of a parallel vector field along an immersed curve in terms of initial data: its value at a single point. The generalization, Theorem B (p), states that a parallel TBI along an immersion $N^p \to M$ is (locally) uniquely determined by initial data of two kinds: one is its value along N^{p-1}, any codimension 1 submanifold of N^p, the other is its value on any ($p-1$)-dimensional distribution which is in "general position" with respect to N^{p-1}. See §5.4.

Since Theorems A (p) and B (p) both depend on the Cartan-Kahler Theorem, a procedure is developed to handle both in a uniform manner. For this purpose, the Cartan-Kahler Theorem, which is usually stated in terms of ideals of real-valued forms, is restated in §1 in terms of differential forms with values in a vector bundle. In §2, the Levi-Civita connection on M is used to obtain certain useful vector bundle valued differential forms on the Grassmann manifolds $\mathscr{G}^p(M)$ and $\mathscr{G}_0^p(M)$. The restrictions of these forms to the vertical subbundles of the tangent bundles $T\mathscr{G}^p(M)$ and $T\mathscr{G}_0^p(M)$ yield important isomorphisms. In §3.3, the conditions for the parallelism of a TBI are restated in terms of the vanishing of a vector bundle valued differential form. The proofs of Theorems A (p) and B (p) then proceed in §§4 and 5, in a like manner. First, in §4.1 and §5.2, the desired solutions are shown to be special integral manifolds of certain vector bundle valued forms on appropriate bundles. Next, in §4.2 and §5.3, the conditions for the unique integrability of these forms using Cartan-Kahler (the existence of regular integral planes and the dimension of their polar spaces) are computed using the isomorphisms obtained in §2. Finally, in §4.3 and §5.4, it is shown that the given initial data satisfies these conditions and hence can be integrated to unique local solutions.

For the C^∞ version of Theorem B (p) proved under somewhat more specialized hypotheses, see [7]. A global C^∞ version of Theorem B (p) appears in [8]; it is not known if the global real analytic version is true.

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1. **Differential forms and the Cartan-Kahler theorem.**

1.1 *Fiber bundles.* Notation follows Bourbaki [3], with the exceptions noted below. "Smooth" will mean real analytic or infinitely differentiable. If (B, M, π) is a smooth fiber bundle with base manifold M, total space B and projection π, then the fiber $\pi^{-1}(x)$, $x \in M$, is denoted $B(x)$ with the exception that in the case of the tangent bundle $T(M)$ the fiber at x will be denoted M_x. When $f\colon N \to M$ is a smooth map of manifolds, $Tf\colon T(N) \to T(M)$ is the tangent map. $f^*B = N \times_M B = \{(x, b) \in N \times B \mid f(x) = \pi(b)\}$ is the *reciprocal image of B under f*. f^*B is a smooth bundle over N whose bundle projection is projection on the first factor; by abuse of notation this map is also denoted π. Projection on the second factor, $f_*\colon f^*B \to B$, is a smooth f-bundle map. If B is a principal bundle (vector bundle) then so is f^*B and f_* is a principal bundle (vector bundle) map. If ω is a smooth section of B there is a unique smooth section $f^*\omega$ of f^*B such that $f_* \circ f^*\omega = \omega \circ f$. If $h\colon E \to F$ is an f-vector bundle map then there is a unique N-vector bundle map $\tilde{h}\colon E \to f^*F$ such that $f_* \circ \tilde{h} = h$. Furthermore, if F' is a second vector bundle over M and $k\colon F \to F'$ is an M-vector bundle map, then there is a unique N-vector bundle map $f^*k\colon f^*F \to f^*F'$ such that $k \circ f_* = f_* \circ f^*k$. Similar statements hold for principal bundles. Let (B, M, π, G) be a smooth principal bundle with group G. For each $\sigma \in G$ the right action of σ on B is denoted $r_\sigma\colon B \to B$; for $b \in B$ $r_\sigma(b)$ is usually written $b\sigma$. Let G act on the left as diffeomorphisms of the manifold H. Let $B \times^G H$ be the set of equivalence classes of elements of $B \times H$ under the relation: $(b, h) \sim (b', h')$ iff there is $\sigma \in G$ such that $b' = b\sigma$ and $h' = \sigma^{-1}(h)$. The image of $(b, h) \in B \times H$ is written $b \cdot h \in B \times^G H$. $B \times^G H$ is then the total space of a smooth bundle over M whose projection is the map $b \cdot h \mapsto \pi(b)$; $B \times^G H$ is the *bundle associated to B with fiber type H* (and given action). In addition $B \times H$ is a principal G bundle over $B \times^G H$ whose projection is $(b, h) \mapsto b \cdot h$. For fixed $b \in B$ the map $\theta_b\colon H \to B \times^G H$ where $\theta_b(h) = b \cdot h$ is a diffeomorphism of H onto $B \times^G H(\pi(b))$. With f as above, $f^*(B \times^G H)$ and $(f^*B) \times^G H$ are canonically bundle isomorphic; $(x, b \cdot h) \mapsto (x, b) \cdot h$. If $B' \subset B$ is a principal subbundle that reduces the group G to a subgroup, then corresponding associated bundles are canonically isomorphic: $B' \times^G H \approx B \times^G H$.

If (E, M, π) is a vector bundle, $\Lambda^p(E)$ is its pth *exterior power*; however, when $E = T(M)$, $\Lambda^p(T(M))$ is abbreviated to $\Lambda^p(M)$. The pth exterior power of a vector bundle map $h: E \to F$ will be denoted $\Lambda^p(h): \Lambda^p(E) \to \Lambda^p(F)$. Only real vector bundles will be considered.

1.2 *Differential forms.* Let (F, M, π) be a smooth vector bundle, over M. A *differential p form on M with values in F* is a smooth M-vector bundle map $\omega: \Lambda^p(M) \to F$. The set of differential p forms on M with values in F is thus in one-to-one correspondence with the smooth sections of the vector bundle Hom $(\Lambda^p(M), F)$. Recall that for finite-dimensional vector spaces V and W there is a canonical isomorphism between Hom $(\Lambda^p(V), W)$ and the space of alternating p-linear maps of V into W. Applying this to vector bundles E and F over M yields a canonical M-vector bundle isomorphism between Hom $(\Lambda^p(E), F)$ and $\mathrm{Alt}^p(E, F)$. Consequently, if ω is a p form on M with values in F, then for each $x \in M$, $\omega|_{\Lambda^p(M)(x)}$ may be considered either as a linear map $\Lambda^p(M)(x) \to F(x)$ or as an alternating p linear map $M_x \to F(x)$. If $F = M \times V$, the space of differential p forms on M with values in F will be identified (by composition with projection on the second factor) with the space of differential p forms on M with values in the vector space V. When $V = \boldsymbol{R}$, these are, of course, the real-valued p forms on M.

Let $F_1, \ldots, F_l$ also be smooth vector bundles over M and $\varphi: F_1 \oplus \cdots \oplus F_l \to F$ a smooth l multilinear M-map. Suppose that for each i $(1 \leq i \leq l)$, ω_i is a p_i-form on M with values in F_i, then there is a unique $p = \sum_{i=1}^{l} p_i$ form $\omega_1 \wedge_\varphi \cdots \wedge_\varphi \omega_l$ on M with values in F called the *exterior product of the ω_i with respect to φ*. $\omega_1 \wedge_\varphi \cdots \wedge_\varphi \omega_l$ is defined by the formula: for $v_1, \ldots, v_p \in M_x$,

$$\omega_1 \wedge_\varphi \cdots \wedge_\varphi \omega_l(v_1, \ldots, v_{p_1}, v_{p_1+1}, \ldots, v_{p_1+p_2}, \ldots, \ldots, v_p)$$

$$(1.2.1) = \sum_\sigma \operatorname{sgn}(\sigma)\varphi[\omega_1(v_{\sigma(1)}, \ldots, v_{\sigma(p_1)}), \omega_2(v_{\sigma(p_1+1)}, \ldots, v_{\sigma(p_1+p_2)}), \ldots, \omega_l(\ldots v_{\sigma(p)})]$$

where σ runs over the $(p_1, p_2, \ldots, p_l)$ shuffle permutations of $\{1, 2, \ldots, p\}$, that is, if $0 \leq n \leq l-1$ and (using $p_0 = 0$) $\sum_{i=0}^{n} p_i < i < j \leq \sum_{i=0}^{n+1} p_i$, then $\sigma(i) < \sigma(j)$. In particular, if ω is a 1 form with values in F, then $\Lambda^p\omega$ is a p form with values in $\Lambda^p(F)$ and

$$\Lambda^p\omega(v_1, \ldots, v_p) = p!\, \omega(v_1) \wedge \cdots \wedge \omega(v_p).$$

Let $f: N \to M$ be smooth and ω be a p form on M with values in F. Let $h = \omega \circ \Lambda^p(Tf): \Lambda^p(N) \to F$. Then $\tilde{h}$ (see 1.1) is a p form on N with values in f^*F. $\tilde{h}$ will be denoted by $\delta f\omega$ and called the *pull back of ω under f*; $f_* \circ \delta f\omega = \omega \circ \Lambda^p(Tf)$. Pulling back commutes with exterior multiplication (for example, if $\varphi: F_1 \oplus F_2 \to F$ is as above, $f^*\varphi: f^*F_1 \oplus f^*F_2 \to f^*F$ and $\delta f(\omega_1 \wedge_\varphi \omega_2) = \delta f\omega_1 \wedge_{f^*\varphi} \delta f\omega_2)$. Pulling back is contravariant with respective to composition (if $g: M \to \tilde{M}$ is smooth, $\delta(g \circ f)\omega = \delta f(\delta g\omega))$.

Let (B, M, π, G) be a smooth principal G bundle and $\rho: G \to \mathrm{Aut}\, V$ a representation of the group G on the finite-dimensional vector space V. Recall that a p form ω on B with values in V is *ρ-equivariant* if $\delta r_\sigma\omega = \rho(\sigma^{-1}) \circ \omega$ for each $\sigma \in G$. ω is *π-horizontal* if $\omega(v_1, \ldots, v_p) = 0$ whenever some v_i is vertical (that is $T\pi(v_i) = 0$). Let $E = B \times^G V$ be the vector bundle over M associated to B by ρ. If ω is a ρ-

equivariant, π-horizontal p form on B with values in V, then $\omega^\#$, the *associated form of* ω, is a p form on M with values in E defined as follows. If $x \in M$ and $v_1, \ldots, v_p \in M_x$, choose $b \in B(x)$ and $\bar{v}_1, \ldots, \bar{v}_p \in B_b$ such that $T\pi\bar{v}_i = v_i$, $i = 1, \ldots, p$, and put

$$\omega^\#(v_1, \ldots, v_p) = b \cdot \omega(\bar{v}_1, \ldots, \bar{v}_p). \tag{1.2.2}$$

$\omega^\#$ is a smooth form; for details see [6, pp. 98–99]. There is a canonical B-isomorphism

$$\pi^*E \approx B \times V, \qquad (b, b \cdot v) \mapsto (b, v). \tag{1.2.3}$$

The natural map π_*: $\pi^*E \to E$, $(b, e) \mapsto e$ is the composition of this isomorphism with the bundle projection $B \times V \to E$, $(b, v) \mapsto b \cdot v$. When π^*E is identified with $B \times V$ by means of (1.2.3) then

$$\delta\pi(\omega^\#) = \omega. \tag{1.2.4}$$

If f: $N \to M$ is smooth then $f_*: f^*B \to B$ and

$$\delta f(\omega^\#) = (\delta f_* \omega)^\#. \tag{1.2.5}$$

1.3 *Differential ideals.* Recall that an ideal in the ring of real-valued differential forms on M is a *differential ideal* if it is finitely generated and closed under the operations of exterior differentiation and projection into the homogeneous forms of the various degrees. It will be assumed that there are no 0 forms (that is, functions) in the differential ideals discussed here. If $\tau_1, \ldots, \tau_k$ are a finite collection of homogeneous forms on M, the ideal $I(\tau_1, \ldots, \tau_k)$ generated by $\tau_1, \ldots, \tau_k$ and their differentials $d\tau_1, \ldots, d\tau_k$ is a differential ideal, said to be the *differential ideal generated by* $\tau_1, \ldots, \tau_k$. Let I be a differential ideal on M. If $E^p \subset M_x$ is a p-dimensional subspace, then E^p is an *integral plane of* I if $\tau|_{E^p} = 0$ for all $\tau \in I$. A submanifold (N, f) of M is an *integral manifold* of I if $\delta f\tau$ vanishes on N for all $\tau \in I$ (equivalently if $Tf(N_x)$ is an integral plane of I for all $x \in N$). The collection $I^p(M)$ of all integral p planes of I is topologized as a subspace of the Grassmann manifold of p planes $\mathcal{G}^p(M)$. When $p = 0$, $E^0 \subset M_x$ is identified with x, so by the standing assumption $I^0(M) = M$. If $E^p \subset M_x$, the *polar space of* E^p *with respect to* I, $H(E^p, I)$, is the subspace of M_x annihilated by all linear functionals of the form $i(v_1 \wedge \cdots \wedge v_q)\tau$ where $v_1, \ldots, v_q \in E^p$, τ is a $q+1$ form in I and $0 \leqq q \leqq p$. Clearly $E^p \in I^p(M)$ if and only if $E^p \subset H(E^p, I)$. If $E^{p-1} \subset E^p$, then $E^p \in I^p(M)$ if and only if $E^p \subset H(E^{p-1}, I)$. E^0 is a *regular* 0 plane if the dimension of $H(\tilde{E}^0, I)$ is constant for $\tilde{E}^0$ in a neighborhood of E^0. An integral plane E^p is *regular* if it contains a regular integral $p-1$ plane and the dimension of $H(\tilde{E}^p, I)$ is constant for $\tilde{E}^p$ in some neighborhood of E^p in $I^p(M)$.

For each $i = 1, \ldots, l$ let τ_i be an s_i form on M with values in the vector bundle (F, M, π). Suppose that F is associated to a principal bundle over M by the action of its group as automorphisms of the k-dimensional vector space V. Let $f_1, \ldots, f_k$

be any basis of V^*. For $x \in M$ choose a vector chart $\varphi\colon \pi^{-1}(\mathscr{U}_\varphi) \to \mathscr{U}_\varphi \times V$ over some neighborhood $\mathscr{U}_\varphi$ of x. Define real-valued s_i forms τ_{ij} on $\mathscr{U}_\varphi$ by

$$\tau_{ij} = f_j \circ (\text{projection on } V) \circ \varphi \circ \tau_i, \qquad 1 \leqq i \leqq l,\ 1 \leqq j \leqq k.$$

Then $I(\tau_{ij}) \equiv I(\tau_{11}, \ldots, \tau_{lk})$ is a differential ideal defined on $\mathscr{U}_\varphi$. If the forms τ'_{ij} are similarly defined using a chart φ' over $\mathscr{U}_{\varphi'}$, it is easily verified that $I(\tau_{ij}) = I(\tau'_{ij})$ over $\mathscr{U}_\varphi \cap \mathscr{U}_{\varphi'}$. In particular, for each $x \in M$, $\tau_1, \ldots, \tau_l$ define an ideal $I(\tau_1, \ldots, \tau_l, x) \equiv I(\tau_{ij})|_{M_x}$ in the Grassmann algebra $\Lambda(M_x^*)$. $E^p \subset M_x$ is an *integral plane of* $(\tau_1, \ldots, \tau_l)$ if it is an integral plane for some $I(\tau_{ij})$ (equivalently if $I(\tau_1, \ldots, \tau_l, x)|_{E^p} = 0$). (N^p, f) *is an integral manifold of* $(\tau_1, \ldots, \tau_l)$ if $\delta f \tau_1, \ldots, \delta f \tau_l$ vanish on N^p (equivalently, if $Tf(N_x^p)$ is an integral plane of $(\tau_1, \ldots, \tau_l)$ for each $x \in N^p$). Let $H(E^p, (\tau_1, \ldots, \tau_l)) \equiv H(E^p, I(\tau_{ij}))$ and define an integral plane E^p of $(\tau_1, \ldots, \tau_l)$ to be *regular* by replacing $H(\tilde{E}^p, I)$ with $H(\tilde{E}^p, (\tau_1, \ldots, \tau_l))$ in the definition above (equivalently, by requiring E^p to be a regular integral plane of some $I(\tau_{ij})$).

1.4 *The Cartan-Kahler Theorem.* A proof of the following theorem appears in [5, pp. 19–28]. It is a preliminary form of the Cartan-Kahler Theorem.

THEOREM 1.4.1. *Let I be a differential ideal on the manifold M. Suppose that $(N^{p-1}$, inclusion) is a $(p-1)$-dimensional integral manifold of I whose tangent plane N_x^{p-1} at x is a regular integral plane with dimension of $H(N_x^{p-1}, I)$ equal to p.*

Then, assuming all the data is real analytic, there is a neighborhood $\mathscr{U}$ of x in M and a unique real analytic p-dimensional integral manifold $(N^p$, inclusion) of I passing through x, contained in $\mathscr{U}$ and containing the connected component of $N^{p-1} \cap \mathscr{U}$ through x. Moreover, $N_x^p = H(N_x^{p-1}, I)$.

In view of the preceding discussion this theorem remains valid when the differential ideal I is replaced by a finite set $(\tau_1, \ldots, \tau_l)$ of differential forms on M with values in a vector bundle F. In fact, work within a single neighborhood $\mathscr{U}_\varphi$ of x and translate from $(\tau_1, \ldots, \tau_l)$ to $I \equiv I(\tau_{ij})$.

2. Standard spaces, bundles and isomorphisms.

2.1 *Standard spaces.* Let Euclidean space, R^m, be equipped with its standard inner product in which $r_1 = (1, \ldots, 0), \ldots, r_m = (0, \ldots, 0, 1)$ is an orthonormal basis. Via $r_1, \ldots, r_m$, $O(m)$, the group of isometries of R^m, is identified with the group of $m \times m$ orthogonal matrices. $r_1 \wedge \cdots \wedge r_m \in \Lambda^m(R^m)$ is the standard orientation of R^m. The subgroup of orientation preserving isometries, $SO(m)$, is identified with the orthogonal matrices of determinant one. Finally, the Lie algebra $o(m)$ of $O(m)$, that is, the set of skew adjoint transformations of R^m, is identified with the skew symmetric $m \times m$ matrices.

For any p, $0 < p < m$, the subspace span $\{r_1, \ldots, r_p\}$ of R^m is identified with R^p and the subspace span $\{r_{p+1}, \ldots, r_m\}$ is identified with R^{m-p} in the obvious manner.

This yields an orthogonal direct sum decomposition

$$R^m = R^p \oplus R^{m-p}, \qquad R^p \subset R^m,\ R^{m-p} \subset R^m.$$

$O(p)$ and $SO(p)$ are then identified with subgroups of $O(m)$ which act as the identity on R^{m-p}, while $O(m-p)$ is identified with the subgroup that acts as the identity on R^p. In particular, using the matrix notation:

$$G \times O(m-p) \equiv \left(\begin{array}{c|c} G & 0 \\ \hline 0 & O(m-p) \end{array}\right) \subset O(m), \qquad G = O(p) \quad \text{or} \quad SO(p).$$

$o(p)$ is identified with the subalgebra of $o(m)$ whose members annihilate R^{m-p}, $o(m-p)$ is identified with the subalgebra whose members annihilate R^p and $\mathrm{Hom}\,(R^p, R^{m-p})$ is identified with the subspace whose members carry R^p into R^{m-p} and R^{m-p} into R^p. Together these spaces yield a direct sum decomposition

$$o(m) = o(p) \oplus o(m-p) \oplus \mathrm{Hom}\,(R^p, R^{m-p}) \tag{2.1.1}$$

which in matrices is the correspondence

$$\begin{pmatrix} A & -C^t \\ C & B \end{pmatrix} \leftrightarrow (A, B, C).$$

Similarly, if $0<q<p<m$, $R^m = R^q \oplus R^{p-q} \oplus R^{m-p}$. Set

$$\Gamma(q,p) = O(q) \times O(p-q) \times O(m-p) \subset O(m)$$

and

$$S\ \Gamma(q,p) = \Gamma(q,p) \cap [SO(p) \times O(m-p)] \subset O(m).$$

Furthermore, $o(q) \oplus o(m-q)$ has a direct sum decomposition

$$o(q) \oplus o(m-q) = o(q) \oplus o(p-q) \oplus o(m-p) \oplus \mathrm{Hom}\,(R^{p-q}, R^{m-p}) \tag{2.1.2}$$

which in matrices is the correspondence

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B & -D^t \\ 0 & D & C \end{pmatrix} \leftrightarrow (A, B, C, D).$$

If V and W are vector spaces and $U \subset V$ is a subspace, then $\mathrm{Hom}\,(U; V, W)$ denotes the subspace of $\mathrm{Hom}\,(V, W)$ consisting of the homomorphisms whose kernels contain U. In particular, there is a direct sum decomposition

$$\mathrm{Hom}\,(R^p, R^{m-p}) = \mathrm{Hom}\,(R^{p-q}; R^p, R^{m-p}) \oplus \mathrm{Hom}\,(R^q; R^p, R^{m-p}). \tag{2.1.3}$$

Here $T \mapsto (T_q, T_{p-q})$ where $T_k = T \circ$ (projection on R^k), $k=q$ or $p-q$. If $o(q) \oplus o(m-q)$ is included in $o(m)$ then using the decompositions 2.1.2 and 2.1.1, $\mathrm{Hom}\,(R^{p-q}, R^{m-p})$ is included in $\mathrm{Hom}\,(R^p, R^{m-p})$ as the subspace

$$\mathrm{Hom}\,(R^q; R^p, R^{m-p}).$$

Suppose $\chi\colon K\to \mathrm{Aut}(V)$ and $\lambda\colon K\to \mathrm{Aut}(W)$ are representations of the group K as automorphisms of the vector spaces V and W.

$\chi\oplus\lambda\colon K\to \mathrm{Aut}(V\oplus W)$ is defined to be the representation such that $(\chi\oplus\lambda)(k)=\chi(k)\oplus\lambda(k)$.

$\mathrm{Hom}(\chi,\lambda)\colon K\to \mathrm{Aut}(\mathrm{Hom}(V,W))$ is defined to be the representation such that $\mathrm{Hom}(\chi,\lambda)(k)T=\lambda(k)\circ T\circ\chi(k^{-1})$ for each $T\in\mathrm{Hom}(V,W)$.

Let G be $O(p)$ or $SO(p)$ in the following two paragraphs.

$\mathrm{I}\colon O(m)\to O(m)$ is the *identity representation*. Its restriction to $G\times O(m-p)$ preserves the splitting $R^m=R^p\oplus R^{m-p}$ and hence decomposes into a direct sum $\rho_p\oplus\rho_{m-p}$ of subrepresentations where

$$\rho_p\colon G\times O(m-p)\to G \quad\text{and}\quad \rho_{m-p}\colon G\times O(m-p)\to O(m-p)$$

are the projections.

$\mathrm{Ad}^m\colon O(m)\to\mathrm{Aut}(o(m))$ is the *adjoint representation*; $\mathrm{Ad}^m=\mathrm{Hom}(I,I)$. Its restriction to $G\times O(m-p)$ preserves the splitting $o(m)=o(p)\oplus o(m-p)\oplus \mathrm{Hom}(R^p,R^{m-p})$ and decomposes into a direct sum $\mathrm{Ad}^p\circ\rho_p\oplus\mathrm{Ad}^{m-p}\circ\rho_{m-p}\oplus\mathrm{Hom}(\rho_p,\rho_{m-p})$ of subrepresentations. In fact, if $\sigma=(\sigma_p,\sigma_{m-p})\in G\times O(m-p)$,

$$\begin{aligned}\mathrm{Ad}^m(\sigma)(A,B,C) &= \begin{pmatrix}\sigma_p & 0\\ 0 & \sigma_{m-p}\end{pmatrix}\begin{pmatrix}A & -C^t\\ C & B\end{pmatrix}\begin{pmatrix}\sigma_p^{-1} & 0\\ 0 & \sigma_{m-p}^{-1}\end{pmatrix}\\ &= (\mathrm{Ad}^p\circ\rho_p(\sigma)A,\ \mathrm{Ad}^{m-p}\circ\rho_{m-p}(\sigma)B,\ \mathrm{Hom}(\rho_p,\rho_{m-p})(\sigma)C).\end{aligned}$$

Thus, for the subgroups $G\times O(m-p)$:

(2.1.4a) $$I=\rho_p\oplus\rho_{m-p}$$

and

(2.1.4b) $$\mathrm{Ad}^m=\mathrm{Ad}^p\circ\rho_p\oplus\mathrm{Ad}^{m-p}\circ\rho_{m-p}\oplus\mathrm{Hom}(\rho_p,\rho_{m-p}).$$

Let G denote $\Gamma(q,p)$ or $S\Gamma(q,p)$ in this paragraph. The restriction of ρ_p to G preserves the splitting $R^p=R^q\oplus R^{p-q}$ and so decomposes into the direct sum of subrepresentations

(2.1.4c) $$\rho_p=\rho_q\oplus\rho_{p-q}$$

where ρ_q and ρ_{p-q} are the projections of G into $O(q)$ and $O(p-q)$ respectively. The restriction of the representation $\mathrm{Hom}(\rho_p,\rho_{m-p})$ to G preserves the splitting 2.1.3 and decomposes into the direct sum of subrepresentations

(2.1.4d) $$\mathrm{Hom}(\rho_p,\rho_{m-p})=\mathrm{Hom}(\rho_q\oplus J_{p-q},\rho_{m-p})\oplus\mathrm{Hom}(J_q\oplus\rho_{p-q},\rho_{m-p})$$

where $J_k\colon G\to O(k)$, $k=q$ or $p-q$, is the trivial representation.

The homogeneous spaces

1. $\mathscr{G}^p(R^m)=O(m)/[O(p)\times O(m-p)]$,
2. $\mathscr{G}_0^p(R^m)=O(m)/[SO(p)\times O(m-p)]$,
3. $\mathscr{G}^p(R^m;R^q)=[O(q)\times O(m-q)]/\Gamma(q,p)$,
4. $\mathscr{G}_0^p(R^m;R^q)=[O(q)\times O(m-q)]/S\Gamma(q,p)$, $q\leq p$,

are Grassman manifolds; in each of them the coset of σ is identified with the p plane $P=\sigma(R^p)$. In the second and fourth, P is an oriented plane, P^+, where $+=\sigma(r_1)\wedge\cdots\wedge\sigma(r_p)=\Lambda^p(\sigma)(r_1\wedge\cdots\wedge r_p)\in\Lambda^p(P)$. In cases three and four, P contains the subspace $R^q=\sigma(R^q)$.

$$\dim \mathscr{G}^p(R^m) = \dim \mathscr{G}_0^p(R^m) = p(m-p);$$
$$\dim \mathscr{G}^p(R^m; R^q) = \dim \mathscr{G}_0^p(R^m; R^q) = (p-q)(m-p).$$

The inclusion map $O(q)\times O(m-q)\to O(m)$ induces smooth maps

$$\mathscr{G}^p(R^m; R^q)\to\mathscr{G}^p(R^m) \quad\text{and}\quad \mathscr{G}_0^p(R^m; R^q)\to\mathscr{G}_0^p(R^m).$$

These maps embed their domains homeomorphically as submanifolds of their ranges, [4, p. 115, Proposition 4.4]. Under the interpretation, above, of the elements of the homogeneous spaces as p planes in a fixed vector space R^m, these embeddings may be taken to be inclusions. Consequently, the two diagrams

(2.1.5)
$$\begin{tikzcd} O(q)\times O(m-q) \arrow[r, "i"] \arrow[d] & O(m) \arrow[d] \\ \mathscr{G}^p(R^m; R^q) \arrow[r, "i"] & \mathscr{G}^p(R^m) \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} O(q)\times O(m-q) \arrow[r, "i"] \arrow[d] & O(m) \arrow[d] \\ \mathscr{G}_0^p(R^m; R^q) \arrow[r, "i"] & \mathscr{G}_0^p(R^m) \end{tikzcd}$$

commute where the vertical maps are bundle projections. Thus at the identity of $O(q)\times O(m-q)$, there are commutative diagrams:

(2.1.6)
$$\begin{tikzcd} o(q)\oplus o(m-q) \arrow[r, "i"] \arrow[d] & o(m) \arrow[d] \\ \mathscr{G}^p(R^m; R^q)_{R^p} \arrow[r, "i"] & \mathscr{G}^p(R^m)_{R^p} \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} o(q)\oplus o(m-q) \arrow[r, "i"] \arrow[d] & o(m) \arrow[d] \\ \mathscr{G}_0^p(R^m; R^q)_{R^{p+}} \arrow[r, "i"] & \mathscr{G}_0^p(R^m)_{R^{p+}} \end{tikzcd}$$

In (2.1.6) each left-hand projection annihilates the vertical subspace $o(q)\oplus o(p-q)\oplus o(m-q)\oplus 0$ and thus maps Hom (R^{p-q}, R^{m-p}) isomorphically onto its image. Similarly the right-hand projection annihilates $o(p)\oplus o(m-p)\oplus 0$ and maps Hom (R^p, R^{m-p}) isomorphically onto its image.

PROPOSITION 2.1.1. *There are canonical isomorphisms*

$$\mathscr{G}^p(R^m, R^q)_{R^p}\approx \operatorname{Hom}(R^{p-q}, R^{m-p})$$

and

$$\mathscr{G}^p(R^m)_{R^p}\approx \operatorname{Hom}(R^p, R^{m-p}).$$

When $\mathscr{G}^p(R^m; R^q)$ is included in $\mathscr{G}^p(R^m)$ as a submanifold, $\mathscr{G}^p(R^m; R^q)_{R^p}$ is carried onto the subspace Hom $(R^q; R^p, R^{m-p})$ *of $\mathscr{G}^p(R^m)_{R^p}$ by the transformation that sends $T\colon R^{p-q}\to R^{m-p}$ into $0\oplus T\colon R^q\oplus R^{p-q}=R^p\to R^{m-p}$. The same result holds for the oriented Grassmannians $\mathscr{G}_0^p(R^m; R^q)$ and $\mathscr{G}_0^p(R^m)$ at R^{p+}.*

Proof. The isomorphisms are those that arise from diagram (2.1.6). The transformation $\mathcal{G}^p(R^m;\ R^q)_{R^p} \to \mathcal{G}^p(R^m)_{R^p}$ is found by using the decompositions (2.1.1), (2.1.2) and (2.1.3) in the diagram (2.1.6). Q.E.D.

2.2 *Standard bundles.* Let $(F(M), M, \pi_2, O(m))$ be the bundle of frames of the Riemannian manifold M; $\dim M=m$. An element $b \in F(M)$ will be considered interchangeably as the isometry $b\colon R^m \to M_{\pi_2(b)}$ or as the frame

$$\{e_1, \ldots, e_m \mid e_i = b(r_i),\ i = 1, \ldots, m\}.$$

The associated bundle

$$(\mathcal{G}^p(M) = F(M) \times^{O(m)} \mathcal{G}^p(R^m),\ M,\ p_1)$$

is the *Grassmann manifold of p planes tangent to M.* $b \cdot P$ is considered as the plane $b(P) \subset M_{\pi_2(b)}$. There is a principal fibration $(F(M), \mathcal{G}^p(M), \pi_1, O(p)\times O(m-p))$ where $\pi_1(b)=b(R^p)$. The associated bundle

$$(\mathcal{G}_0^p(M) = F(M) \times^{O(m)} \mathcal{G}_0^p(R^m),\ M,\ p_0)$$

is the *Grassmann manifold of oriented p planes tangent to M.* $b \cdot P^+$ is considered as the plane $b(P) \subset M_{\pi_2(b)}$ with the orientation it inherits from P^+ via b. There is a principal fibration $(F(M), \mathcal{G}_0^p(M), \pi_0, SO(p)\times O(m-p))$ where $\pi_0(b)=b(R^p)^+$ and $+=b(r_1)\wedge \cdots \wedge b(r_p)$. The opposite orientation to $+$ will be denoted $-$. There is a principal fibration $(\mathcal{G}_0^p(M), \mathcal{G}^p(M), z, O(1))$ where $z(P^\pm)=P$. This fibration is preserved by the *antipodal map* $A_0\colon \mathcal{G}_0^p(M) \to \mathcal{G}_0^p(M)$ where $A_0(P^\pm)=P^\mp$. The bundle projections are summarized in the commutative diagram:

(2.2.1)
$$\begin{tikzcd}
& F(M) \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] \arrow[dd, "\pi_2"] & \\
\mathcal{G}_0^p(M) \arrow[rr, "z"] \arrow[dr, "p_0"'] & & \mathcal{G}^p(M) \arrow[dl, "p_1"] \\
& M &
\end{tikzcd}$$

Let

$$\begin{aligned}
&(\mathcal{G}^T = F(M) \times^{O(p)\times O(m-p)} R^p,\ \mathcal{G}^p(M),\ \text{projection}),\\
&(\mathcal{G}^\perp = F(M) \times^{O(p)\times O(m-p)} R^{m-p},\ \mathcal{G}^p(M),\ \text{projection}),\\
&(\mathcal{G}_0^T = F(M) \times^{SO(p)\times O(m-p)} R^p,\ \mathcal{G}_0^p(M),\ \text{projection}) \text{ and}\\
&(\mathcal{G}_0^\perp = F(M) \times^{SO(p)\times O(m-p)} R^{m-p},\ \mathcal{G}_0^p(M),\ \text{projection})
\end{aligned}$$

be the vector bundles associated to $F(M)$ by the representations ρ_p, ρ_{m-p}, $\rho_p|_{SO(p)\times O(m-p)}$ and $\rho_{m-p}|_{SO(p)\times O(m-p)}$ respectively. The bundles $\mathcal{G}^T$ and $\mathcal{G}_0^T$ are called the *canonical vector bundles* and the bundles $\mathcal{G}^\perp$ and $\mathcal{G}_0^\perp$ are called the *canonical normal bundles.* These names are used for the following reason. If $b \in F(M)$, let $x=\pi_2(b)$, $P=b(R^p)=\pi_1(b)$, $P^+=\pi_0(b)$ where $+=\Lambda^p(b)(r_1\wedge \cdots \wedge r_p)$, and $\perp P \subset M_x$ be the orthogonal complement of $P \subset M_x$. If r belongs to R^p or R^{m-p} (as the case may be) consider $b \cdot r$ as $b(r) \in M_x$. Then the fibers $\mathcal{G}^T(\pi_1(b))$ and $\mathcal{G}_0^T(\pi_0(b))$ are identified with P and the fibers $\mathcal{G}^\perp(\pi_1(b))$ and $\mathcal{G}_0^\perp(\pi_0(b))$ are identified with $\perp P$.

These four bundles are thus Euclidean vector bundles. That is, the fibers are equipped with inner products that vary smoothly with the base point; $\langle b\cdot r, b\cdot s\rangle = \langle b(r), b(s)\rangle = \langle r, s\rangle$.

Hom $(\mathscr{G}^T, \mathscr{G}^\perp)$ and Hom $(\mathscr{G}_0^T, \mathscr{G}_0^\perp)$ are the vector bundles associated to $F(M)$ by the representations Hom (ρ_p, ρ_{m-p}) and Hom $(\rho_p, \rho_{m-p})|_{SO(p)\times O(m-p)}$ respectively.

$\mathscr{G}^T \oplus \mathscr{G}^\perp$ and $\mathscr{G}_0^T \oplus \mathscr{G}_0^\perp$ are the vector bundles associated to $F(M)$ by the representations $\rho_p \oplus \rho_{m-p} = I|_{O(p)\oplus O(m-p)}$ and $\rho_p \oplus \rho_{m-p}|_{SO(p)\oplus O(m-p)} = I|_{SO(p)\oplus O(m-p)}$ respectively. Since the tangent bundle $T(M) = F(M)\times^{O(m)} R^m$ is the vector bundle over M associated to $F(M)$ by the representation I, so $p_1^*T(M) = p_1^*F(M)\times^{O(m)} R^m$ is the vector bundle over $\mathscr{G}^p(M)$ associated to $p_1^*F(M)$ by the representation I by §1.1. But $F(M)$ is a reduction of the group of $p_1^*F(M)$ (namely $O(m)$) to the subgroup $O(p)\times O(m-p)$. Hence, by §1.1, there is a canonical isomorphism

(2.2.2a) $$\mathscr{G}^T \oplus \mathscr{G}^\perp \approx p_1^*T(M); \qquad b\cdot r + b\cdot s \mapsto (\pi_1(b), b(r+s)).$$

By the same reasoning,

(2.2.2b) $$\mathscr{G}_0^T \oplus \mathscr{G}_0^\perp \approx p_0^*T(M).$$

Similarly,

(2.2.3a) $$\mathscr{G}_0^T \approx z^*\mathscr{G}^T,$$

(2.2.3b) $$\mathscr{G}_0^\perp \approx z^*\mathscr{G}^\perp.$$

The bundle $\mathscr{G}_0^T$ has as additional structure, a canonical orientation, that is, a natural cross section $c\colon \mathscr{G}_0^p(M)\to \Lambda^p(\mathscr{G}_0^T)$ of its p fold exterior product. For $P^+\in \mathscr{G}_0^p(M)$, $\mathscr{G}_0^T(P^+)$ has been identified with P. Define $c(P^+)=+$. A routine computation using a local section of $F(M)$ over $\mathscr{G}_0^p(M)$ shows that c is a smooth section. With respect to the Euclidean structure on $\Lambda^p(\mathscr{G}_0^T)$, c is a unit cross section. For each q, $0<q<p$, c determines a smooth vector bundle map

$$h\colon \Lambda^q(\mathscr{G}_0^T)\to \Lambda^{p-q}(\mathscr{G}_0^T)$$

as follows: if $u\in \Lambda^q(\mathscr{G}_0^T)(P^+)$ and $v\in \Lambda^{p-q}(\mathscr{G}_0^T)(P^+)$, then $\langle h(u), v\rangle = \langle c(P^+), u\wedge v$. In particular, if $q=p-1$ and $v_1\wedge\cdots\wedge v_p = +$, then

(2.2.4) $$h(v_1\wedge\cdots\wedge v_{i-1}\wedge v_{i+1}\wedge\cdots\wedge v_p) = (-1)^{p-i}v_i.$$

2.3 *Standard isomorphisms*. The vertical subspace of the tangent space at a point b of a bundle B will be denoted $V(b)$, however, if B is the total space of several fibrations the different vertical subspaces will be distinguished by superscripting $V(b)$ with the appropriate bundle projections.

Let φ be the one form of the Riemannian connection on $F(M)$; see [2, pp. 129–131]. If $g_b\colon O(m)\to F(M)(\pi_2(b))$ denotes the diffeomorphism $\sigma\mapsto b\sigma$, then φ is an Ad^m-equivariant form with values in $o(m)$ such that

(2.3.1) $$\varphi|_{V^{\pi_2}(b)} = [Tg_b|_{\text{identity}}]^{-1}.$$

φ_p, φ_{m-p} and $\varphi_{p,m-p}$ will denote the composition of φ with, respectively, the first, second and third projections in the decomposition (2.1.1). There is, for each $b \in F(M)$, a commutative diagram

$$\begin{array}{ccc} O(m) & \xrightarrow{g_b} & F(M)(\pi_2(b)) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi_1} \\ \mathscr{G}^p(R^m) & \xrightarrow{\theta_b} & \mathscr{G}^p(M)(\pi_2(b)) \end{array}$$

where π and π_1 are bundle projections and θ_b is the diffeomorphism $P \mapsto b(P)$. At the tangent space to the identity of $O(m)$ there is then a commutative diagram

$$\begin{array}{ccc} o(m) & \xrightarrow{Tg_b} & V^{\pi_2}(b) \\ \Big\downarrow{\scriptstyle T\pi} & & \Big\downarrow{\scriptstyle T\pi_1} \\ \mathscr{G}^p(R^m)_{R^p} & \xrightarrow{T\theta_b} & V^{p_1}(\pi_1(b)). \end{array}$$

As in (2.1.6), $\ker T\pi = o(p) \oplus o(m-p)$ so that

$$V^{\pi_1}(b) = \ker T\pi_1 = Tg_b(o(p) \oplus o(m-p)). \tag{2.3.2}$$

Thus by (2.3.1),

$$\varphi_{p,m-p}|_{V^{\pi_1}(b)} = 0$$

and $\varphi_{p,m-p}$ is a π_1-horizontal form. By (2.1.4b) $\varphi_{p,m-p}$ is $\operatorname{Hom}(\rho_p, \rho_{m-p})$ equivariant. Let $\Phi = \varphi_{p,m-p}{}^{\#}$ (see (1.2.2)) be the associated one form on $\mathscr{G}^p(M)$ with values in $\operatorname{Hom}(\mathscr{G}^T, \mathscr{G}^{\perp})$. $\varphi_{p,m-p}$ is also π_0-horizontal; $\Phi_0 = \varphi_{p,m-p}{}^{\#}$ will denote the corresponding one form on $\mathscr{G}^p_0(M)$ with values in $\operatorname{Hom}(\mathscr{G}^T_0, \mathscr{G}^{\perp}_0)$. Plainly, $\delta z\Phi = \Phi_0$.

Proposition 2.3.1. *The restriction of the form Φ to the vertical subbundle V^{p_1} of the tangent bundle $T(\mathscr{G}^p(M))$ is a $\mathscr{G}^p(M)$-isomorphism of V^{p_1} onto* $\operatorname{Hom}(\mathscr{G}^T, \mathscr{G}^{\perp})$. *In particular, $V^{p_1}(P) \approx \operatorname{Hom}(P, \perp P)$. Similarly, Φ_0 defines a $\mathscr{G}^p_0(M)$-isomorphism of V^{p_0} onto* $\operatorname{Hom}(\mathscr{G}^T_0, \mathscr{G}^{\perp}_0)$ *and* $V^{p_0}(P^+) \approx \operatorname{Hom}(P, \perp P)$.

Proof. Suppose $P = \pi_1(b)$. By (2.3.2)

$$T\pi_1 \colon Tg_b(\operatorname{Hom}(R^p, R^{m-p})) \to V^{p_1}(P)$$

is an isomorphism. Thus,

$$\begin{aligned} \Phi(V^{p_1}(P)) &= \Phi(T\pi_1 \circ Tg_b[\operatorname{Hom}(R^p, R^{m-p})]) \\ &= b\cdot(\varphi_{p,m-p}(Tg_b[\operatorname{Hom}(R^p, R^{m-p})])) \\ &= b\cdot[\operatorname{Hom}(R^p, R^{m-p})] = \operatorname{Hom}(P, \perp P) = \operatorname{Hom}(\mathscr{G}^T, \mathscr{G}^{\perp})(P). \end{aligned}$$

The oriented case proceeds in the same way. Q.E.D.

The isomorphisms in the proposition are in fact independent of the choice of connection since they depend on φ only through the equivariance and (2.3.1); properties shared by all connections.

Let $H=\ker \varphi$ be the horizontal distribution on $F(M)$ associated with φ. Then $K=T\pi_1 H=\ker \Phi$ is a distribution on $\mathscr{G}^p(M)$; as a subbundle of $T(\mathscr{G}^p(M))$ it is a complement to V^{p_1}. K is the *horizontal distribution on* $\mathscr{G}^p(M)$. Similarly, $K_0 =T\pi_0 H=\ker \Phi_0$ is the *horizontal distribution on* $\mathscr{G}_0^p(M)$. By (2.1.4b), $\varphi_p \oplus \varphi_{m-p}$ is $\mathrm{Ad}^m|_{G\times O(m-p)}=\mathrm{Ad}^p \circ \rho_p \oplus \mathrm{Ad}^{m-p} \circ \rho_{m-p}$ equivariant and is the one form of a connection on $F(M)$ over $\mathscr{G}^p(M)$ or $\mathscr{G}_0^p(M)$. The horizontal distribution of this connection will be denoted $\bar{H}$: $\bar{H}(b)=H(b)+Tg_b(\mathrm{Hom}\,(R^p, R^{m-p}))$.

Let ω be the natural one form on $F(M)$; $\omega(b)=b^{-1}\circ T\pi_2$. ω is a π_2 horizontal, I-equivariant form with values in R^m. $\omega^\#$, the associated one form on M with values in $T(M)$ is the identity map. ω_p and ω_{m-p} will denote the compositions of ω with the first and second projections respectively in the decomposition $R^m=R^p \oplus R^{m-p}$. From the first structural equation of the Riemannian connection, $d\omega=-\varphi \wedge \omega$, it follows that

$$d\omega_p = -\varphi_p \wedge \omega_p+(\varphi_{p,m-p})^t \wedge \omega_{m-p} \text{ and} \tag{2.3.3a}$$

$$d\omega_{m-p} = -\varphi_{p,m-p} \wedge \omega_p-\varphi_{m-p} \wedge \omega_{m-p}. \tag{2.3.3b}$$

ω_p and ω_{m-p} are each π_0 and π_1 horizontal; by (2.1.4a), ω_p is ρ_p-equivariant while ω_{m-p} is ρ_{m-p}-equivariant. Let $\omega^T=\omega_p{}^\#$ and $\omega^\perp=\omega_{m-p}{}^\#$ be the associated one forms on $\mathscr{G}^p(M)$ with values in $\mathscr{G}^T$ and $\mathscr{G}^\perp$ respectively. Similarly define $\omega_0^T=\omega_p{}^\#$ and $\omega_0^\perp=\omega_{m-p}{}^\#$ on $\mathscr{G}_0^p(M)$ with values in $\mathscr{G}_0^T$ and $\mathscr{G}_0^\perp$ respectively. Plainly $\omega_0^T=\delta z\omega^T$ and $\omega_0^\perp=\delta z\omega^\perp$. Suppose that $u\in \mathscr{G}^p(M)_P$; choose $b\in F(M)(P)$ and $\bar{u}\in F(M)_b$ such that $T\pi_1(\bar{u})=u$. Then

$$\omega^T|_P(u) = b\cdot(\omega_p(\bar{u})) = b\cdot(\text{projection on } R^p\circ b^{-1}\circ T\pi_2(\bar{u})).$$

So under the identification of $\mathscr{G}^T(P)$ with P for any $P\in \mathscr{G}_0^p(M)$

$$\omega^T|_P = (\text{projection on } P)\circ Tp_1. \tag{2.3.4a}$$

Similarly

$$\omega^\perp|_P = (\text{projection on } \perp P)\circ Tp_1 \tag{2.3.4b}$$

and, for any $P^+\in \mathscr{G}_0^p(M)$

$$\omega_0^T|_{P^+} = (\text{projection on } P)\circ Tp_0, \tag{2.3.5a}$$

$$\omega_0^\perp|_{P^+} = (\text{projection on } \perp P)\circ Tp_0. \tag{2.3.5b}$$

In particular, $\omega^T\oplus\omega^\perp|_K\colon K\to \mathscr{G}^T\oplus\mathscr{G}^\perp$ and $\omega_0^T+\omega_0^\perp|_{K_0}\colon K_0\to \mathscr{G}_0^T\oplus\mathscr{G}_0^\perp$ are $\mathscr{G}^p(M)$ and $\mathscr{G}_0^p(M)$ vector bundle isomorphisms respectively. Thus

$$\omega^T\oplus\omega^\perp\oplus\Phi\colon T(\mathscr{G}^p(M))\to \mathscr{G}^T\oplus\mathscr{G}^\perp\oplus \mathrm{Hom}\,(\mathscr{G}^T, \mathscr{G}^\perp)$$

and

$$\omega_0^T\oplus\omega_0^\perp\oplus\Phi_0\colon T(\mathscr{G}_0^p(M))\to \mathscr{G}_0^T\oplus\mathscr{G}_0^\perp\oplus \mathrm{Hom}\,(\mathscr{G}_0^T, \mathscr{G}_0^\perp)$$

provide direct sum decompositions of the tangent bundles of the Grassman manifolds.

Let D be a smooth q-dimensional ($q \leqq p$) distribution on M. The *bundle of adapted frames of D* is the submanifold $F(D) \subset F(M)$ where

$$F(D) = \{b \in F(M) \mid b(R^q) = D(\pi_2(b))\}.$$

$(F(D), M, \pi_2)$ is a principal fibration that reduces the group of $F(M)$ to $O(q) \times O(m-q)$. Thus $\mathscr{G}^p(M)$ and $F(D) \times^{O(q) \times O(m-q)} \mathscr{G}^p(R^m)$ are canonically isomorphic. The associated bundle

$$(\mathscr{G}^p(D) = F(D) \times^{O(q) \times O(m-q)} \mathscr{G}^p(R^m; R^q), M, p_1)$$

is the *bundle of p-planes containing D*. $b \cdot P \in \mathscr{G}^p(D)$ is considered as the plane $b(P) \subset M_{\pi_2(b)}$ and it contains $b(R^q) = D(\pi_2(b))$. The inclusion $\mathscr{G}^p(R^m; R^q) \to \mathscr{G}^p(R^m)$ of (2.1.5) induces an inclusion i: $\mathscr{G}^p(D) \to \mathscr{G}^p(M)$ as a subbundle over M. The reciprocal image $i^*F(M) = F(M)|\mathscr{G}^p(D)$ is a principal $O(p) \times O(m-p)$ bundle over $\mathscr{G}^p(D)$. $F(D)$ is also a submanifold of $F(M)|\mathscr{G}^p(D)$ where $b \in F(D)$ is identified with $(b(R^p), b) = (P, b) \in F(M)|\mathscr{G}^p(D)$. $(F(D), \mathscr{G}^p(D), \pi_1)$ is a principal fibration that reduces the group of $F(M)|\mathscr{G}^p(D)$ to $\Gamma(q, p)$. In particular, $\mathscr{G}^T|\mathscr{G}^p(D)$ and the vector bundle $F(D) \times^{\Gamma(q,p)} R^p$, associated to $F(D)$ by the restriction of ρ_p to $\Gamma(q, p)$ are canonically isomorphic. Thus, by (2.1.4c) $\mathscr{G}^T|\mathscr{G}^p(D)$ decomposes into a direct sum of subbundles

$$(\mathscr{G}^{T,q} = F(D) \times^{\Gamma(q,p)} R^q, \mathscr{G}^p(D), \text{projection})$$

and

$$(\mathscr{G}^{T,p-q} = F(D) \times^{\Gamma(q,p)} R^{p-q}, \mathscr{G}^p(D), \text{projection})$$

associated to $F(D)$ by the representations ρ_q and ρ_{p-q} respectively. By the identification $\mathscr{G}^T|\mathscr{G}^p(D)(P) = P$, $b \cdot r \mapsto b(r)$, there is a natural isomorphism $\mathscr{G}^{T,q}(P) \approx D(p_1(P))$ and a natural isomorphism $\mathscr{G}^{T,p-q}(P) \approx P \cap \perp D(p_1(P))$. Similarly,

$$\mathrm{Hom}\,(\mathscr{G}^T, \mathscr{G}^\perp)|\mathscr{G}^p(D)$$

and the vector bundle $F(D) \times^{\Gamma(q,p)} \mathrm{Hom}\,(R^p, R^{m-p})$, associated to $F(D)$ by the restriction of $\mathrm{Hom}\,(\rho_p, \rho_{m-p})$ to $\Gamma(q, p)$, are canonically isomorphic. By (2.1.4d) the bundle $\mathrm{Hom}\,(\mathscr{G}^T, \mathscr{G}^\perp)|\mathscr{G}^p(D)$ decomposes into a direct sum of subbundles

$$\mathrm{Hom}\,(\mathscr{G}^{T,p-q}; \mathscr{G}^T|\mathscr{G}^p(D), \mathscr{G}^\perp|\mathscr{G}^p(D)) = F(D) \times^{\Gamma(q,p)} \mathrm{Hom}\,(R^{p-q}; R^p, R^{m-p})$$

and

$$\mathrm{Hom}\,(\mathscr{G}^{T,q}; \mathscr{G}^T|\mathscr{G}^p(D), \mathscr{G}^\perp|\mathscr{G}^p(D)) = F(D) \times^{\Gamma(q,p)} \mathrm{Hom}\,(R^q; R^p, R^{m-p}),$$

associated to $F(D)$ by the representations $\mathrm{Hom}\,(\rho_q \oplus J_{p-q}, \rho_{m-p})$ and $\mathrm{Hom}\,(J_q \oplus \rho_{p-q}, \rho_{m-p})$ respectively. There are natural isomorphisms

$$\mathrm{Hom}\,(\mathscr{G}^{T,p-q}; \mathscr{G}^T|\mathscr{G}^p(D), \mathscr{G}^\perp|\mathscr{G}^p(D))(P) \approx \mathrm{Hom}\,(P \cap \perp D(p_1(P)); P, \perp P),$$
$$\mathrm{Hom}\,(\mathscr{G}^{T,q}; \mathscr{G}^T|\mathscr{G}^p(D), \mathscr{G}^\perp|\mathscr{G}^p(D))(P) \approx \mathrm{Hom}\,(D(p_1(P)); P, \perp P).$$

$(\mathscr{G}^p_0(D), M, p_0)$, the *bundle of oriented p planes containing D*, is constructed in the same way; $(F(D), \mathscr{G}^p_0(D), \pi_0)$ is a principal $S\Gamma(q, p)$ fibration and the vector bundles $\mathscr{G}^{T,q}_0$ and $\mathscr{G}^{T,p-q}_0$ are associated by the representations ρ_q and ρ_{p-q} restricted to $S\Gamma(q, p)$, respectively.

PROPOSITION 2.3.2. *Let D be a smooth q-dimensional distribution on M; $q<p$. When $(\mathscr{G}^p(D), i)$ is considered as a submanifold of $\mathscr{G}^p(M)$, the restriction of $\delta i\Phi$ to the vertical subbundle $V^{p_1,D}$ of the tangent bundle $T\mathscr{G}^p(D)$ is a $\mathscr{G}^p(D)$ isomorphism of $V^{p_1,D}$ onto the subbundle*

$$\mathrm{Hom}\,(\mathscr{G}^{T,q};\ \mathscr{G}^T|\mathscr{G}^p(D),\ \mathscr{G}^\perp|\mathscr{G}^p(D))$$

of $\mathrm{Hom}\,(\mathscr{G}^T, \mathscr{G}^\perp)|\mathscr{G}^p(D)$. *In particular, $V^{p_1,D}(P)\approx\mathrm{Hom}\,(D(p_1(P)); P, \perp P)$. Similarly, $\delta i\Phi_0$ defines a $\mathscr{G}^p_0(D)$ isomorphism of $V^{p_0,D}$ onto*

$$\mathrm{Hom}\,(\mathscr{G}_0^{T,q};\ \mathscr{G}_0^T|\mathscr{G}_0^p(D),\ \mathscr{G}_0^\perp|\mathscr{G}_0^p(D))$$

and $V^{p_0,D}(P^+)\approx\mathrm{Hom}\,(D(p_0(P^+)); P, \perp P)$.

Proof. Let $P\in\mathscr{G}^p(D)$ and choose $b\in F(D)$ so that $\pi_1(b)=P$. The diagram

$$\begin{tikzcd}
& O(q)\times O(m-q) \arrow[r, "i"] \arrow[d, "g_b"] & O(m) \arrow[d, "g_b"] & \\
F(D)\supset & F(D)(\pi_2(b)) \arrow[r, "i"] \arrow[d, "\pi_1"] & F(M)(\pi_2(b)) & \subset F(M) \\
\mathscr{G}^p(D)\supset & \mathscr{G}^p(D)(\pi_2(b)) \arrow[r, "i"] & \mathscr{G}^p(M)(\pi_2(b)) \arrow[from=u, "\pi_1"] & \subset\mathscr{G}^p(M)
\end{tikzcd}$$

commutes, where each i is inclusion, each π_1 is bundle projection and each g_b is the diffeomorphism $\sigma\mapsto b\sigma$. Thus at the identity of $O(q)\times O(m-q)$ there is a commutative diagram

$$\begin{tikzcd}
o(q)\oplus o(m-q) \arrow[r, "i"] \arrow[d, "Tg_b"] & o(m) \arrow[d, "Tg_b"] \\
V^{\pi_2,D}(b) \arrow[r, "Ti"] \arrow[d, "T\pi_1"] & V^{\pi_2}(b) \arrow[d, "T\pi_1"] \\
V^{p_1,D}(P) \arrow[r, "Ti"] & V^{p_1}(P)
\end{tikzcd}$$

where $V^{\pi_2,D}(b)$ is the vertical subspace of $F(D)_b$ as a bundle over M, $V^{p_1,D}(P)$ is the vertical subspace of $\mathscr{G}^p(D)_P$ as a bundle over M, each horizontal map is injective, each $T\pi_1$ is surjective and each Tg_b is bijective. As in the proof of Proposition 2.3.1,

$$T\pi_1\colon Tg_b[\mathrm{Hom}\,(R^p, R^{m-p})]\to V^{p_1}(P)$$

is an isomorphism. Hence, by the commutativity of the preceding diagram and (2.1.2),

$$(*)\qquad T\pi_1\colon Tg_b[\mathrm{Hom}\,(R^{p-q}, R^{m-p})]\to V^{p_1,D}(P)$$

is an isomorphism. Thus

$$\begin{aligned} i_*\delta i\Phi(V^{p_1,D}(P)) &= \Phi(Ti(V^{p_1,D}(P))) = \Phi(T\pi_1\circ Tg_b\circ i(\operatorname{Hom}(R^{p-q}, R^{m-p}))) \\ &= b\cdot(\varphi_{p,m-p}\circ Tg_b(\operatorname{Hom}(R^q; R^p, R^{m-p}))) \\ &= b\cdot(\operatorname{Hom}(R^q; R^p, R^{m-p})) = i_*[b\cdot(\operatorname{Hom}(R^q; R^p, R^{m-p}))] \\ &= i_*[\operatorname{Hom}(\mathscr{G}^{T,q}; \mathscr{G}^T|\mathscr{G}^p(D), \mathscr{G}^\perp|\mathscr{G}^p(D))(P)]. \end{aligned}$$

The proof for the oriented Grassmann manifold proceeds similarly. Q.E.D.

The isomorphisms in the proposition are again independent of the choice of connection. Using (*) it is easy to establish a canonical $\mathscr{G}^p(D)$ isomorphism $V^{p_1,D}\approx \operatorname{Hom}(\mathscr{G}^{T,p-q}, \mathscr{G}^\perp)$ directly. This comment and Propositions 2.3.1 and 2.3.2 may be considered as an extension of Proposition 2.2.1.

Let $f\colon N\to M^m$ be a smooth map and D a smooth q-dimensional distribution along f (that is, $D(x)\subset M_{f(x)}$ for each $x\in N$). Assume that $q<p<m$. The *bundle of adapted frames of D* is the submanifold $F(D)$ of $f^*F(M)$ where

$$F(D) = \{(x, b)\in f^*F(M) \mid b(R^q) = D(x)\}.$$

$(F(D), N, \pi_2)$ is a reduction of $f^*F(M)$ to the subgroup $O(q)\times O(m-q)$ so that $F(D)\times^{O(q)\times O(m-q)}\mathscr{G}^p(R^m)$ and $f^*\mathscr{G}^p(M)$ are canonically isomorphic. ($\mathscr{G}^p(D) = F(D)\times^{O(q)\times O(m-q)}\mathscr{G}^p(R^m; R^q), N, p_1)$ *is the bundle of p planes containing D* and is included, $i\colon \mathscr{G}^p(D)\to f^*\mathscr{G}^p(M)$, as a subbundle over N. $F(D)\subset i^*f^*F(M) = f^*F(M)|\mathscr{G}^p(D)$. $(F(D), \mathscr{G}^p(D), \pi_1)$ is a principal fibration reducing the group of $f^*F(M)|\mathscr{G}^p(D)$, namely $O(p)\times O(m-p)$, to the subgroup $\Gamma(q,p)$. In the manner described above, this reduction induces splittings of the vector bundles

$$(f_*)^*\mathscr{G}^T|\mathscr{G}^p(D) \quad\text{and}\quad (f_*)^*\operatorname{Hom}(\mathscr{G}^T, \mathscr{G}^\perp)|\mathscr{G}^p(D)$$

as follows:

$$(f_*)^*\mathscr{G}^T|\mathscr{G}^p(D) = (f_*)^*\mathscr{G}^{T,q}\oplus(f_*)^*\mathscr{G}^{T,p-q}, \tag{2.3.6a}$$

$$\begin{aligned} &(f_*)^*\operatorname{Hom}(\mathscr{G}^T, \mathscr{G}^\perp)|\mathscr{G}^p(D) \\ &\quad = \operatorname{Hom}[(f_*)^*\mathscr{G}^{T,q}; (f_*)^*\mathscr{G}^T|\mathscr{G}^p(D), (f_*)^*\mathscr{G}^\perp|\mathscr{G}^p(D)] \\ &\quad\quad \oplus \operatorname{Hom}[(f_*)^*\mathscr{G}^{T,p-q}; (f_*)^*\mathscr{G}^T|\mathscr{G}^p(D), (f_*)^*\mathscr{G}^\perp|\mathscr{G}^p(D)] \end{aligned} \tag{2.3.6b}$$

(f_* is the natural map: $f^*\mathscr{G}^p(M)\to\mathscr{G}^p(M)$).

The construction of the vector bundles over the *bundle of oriented p planes containing D*, ($\mathscr{G}^p_0(D) = F(D)\times^{O(q)\times O(m-q)}\mathscr{G}^p_0(R^m; R^q), N, p_0)$, proceeds similarly. The forms Φ and Φ_0 pull back to forms $\delta f_*\Phi$ and $\delta f_*\Phi_0$ on $f^*\mathscr{G}^p(M)$ and $f^*\mathscr{G}^p_0(M)$ respectively and have values in $(f_*)^*\operatorname{Hom}(\mathscr{G}^T, \mathscr{G}^\perp)$ and $(f_*)^*\operatorname{Hom}(\mathscr{G}^T_0, \mathscr{G}^\perp_0)$ respectively.

PROPOSITION 2.3.3. *Let $f\colon N\to M$ be a smooth map and D a smooth q-dimensional ($q<p$) distribution along f.*

The restriction of $\delta f_*\Phi$ to the vertical subbundle V^{p_1} of the tangent bundle $T(f^*\mathscr{G}^p(M))$ is an $f^*\mathscr{G}^p(M)$ isomorphism of V^{p_1} onto $(f_*)^*\operatorname{Hom}(\mathscr{G}^T, \mathscr{G}^\perp)$. *In particular, $V^{p_1}(P)\approx\operatorname{Hom}(P, \perp P)$.*

When $(\mathcal{G}^p(D), i)$ is considered as a submanifold of $f^*\mathcal{G}^p(M)$, the restriction of $\delta(f_* \circ i)\Phi$ to the vertical subbundle $V^{p_1,D}$ of the tangent bundle $T(\mathcal{G}^p(D))$ is an $\mathcal{G}^p(D)$ isomorphism of $V^{p_1,D}$ onto the subbundle

$$\operatorname{Hom}((f_*)^*\mathcal{G}^{T,q}; (f_*)^*\mathcal{G}^T|\mathcal{G}^p(D), (f_*)^*\mathcal{G}^\perp|\mathcal{G}^p(D))$$

of $(f_*)^* \operatorname{Hom}(\mathcal{G}^T, \mathcal{G}^\perp)|\mathcal{G}^p(D)$. *In particular*

$$V^{p_1,D}(P) \approx \operatorname{Hom}(D(p_1(P)); P, \perp P).$$

In the oriented case corresponding isomorphisms are obtained by restricting $\delta(f_* \circ i)\Phi_0$ to vertical subbundles.

The proof of Proposition 2.3.3 is a repetition of the techniques used in the proofs of Propositions 2.3.1 and 2.3.2.

3. **Parallel tangent bundle isometries.**

3.1 *The parallelism of a TBI.* $G: T(N) \to T(M)$.

Let $g: N^p \to M^m$ be a (not necessarily isometric) immersion of Riemannian manifolds. Let E and F be Euclidean vector bundles over N and M respectively and let $G: E \to F$ be a smooth g-map of vector bundles.

Definition. G is a *Euclidean map along* g if it maps fibers $E(x)$ isometrically into fibers $F(g(x))$. If in addition, $E = T(N)$ and $F = T(M)$ then G is a *tangent bundle isometry* (TBI) *along* g.

If $g: N \to M$ is an isometric immersion, then plainly $Tg: T(N) \to T(M)$ is a TBI along g.

A TBI G along g determines a smooth lift $\bar{G}$ of g into $\mathcal{G}^p(M)$ by the formula $\bar{G}(x) = (g(x), G(N_x))$. (Foot points will sometimes be given, for emphasis.) The reciprocal images of bundles over $\mathcal{G}^p(M)$ under $\bar{G}$ determine bundles over N; these are related by a commutative diagram:

(3.1.1)
$$\begin{array}{ccc}
F(G) \equiv \bar{G}^*F(M) & \xrightarrow{\bar{G}_*} & F(M) \\
\downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} \\
\mathcal{G}_0^p(G) \equiv \bar{G}^*\mathcal{G}_0^p(M) & \xrightarrow{\bar{G}_*} & \mathcal{G}_0^p(M) \\
{\scriptstyle z}\downarrow & & \downarrow{\scriptstyle z} \\
N & \xrightarrow{\bar{G}} & \mathcal{G}^p(M) \\
& {\scriptstyle g}\searrow & \downarrow{\scriptstyle p_1} \\
& & M
\end{array}$$

Here $F(G)$ is $F(\operatorname{im} G)$, the bundle of adapted frames of the distribution $\operatorname{im} G$ along g; $\operatorname{im} G(x) = G(N_x) \subset M_{g(x)}$ for each $x \in N$. $\mathcal{G}_0^p(G)$ is $\mathcal{G}_0^p(\operatorname{im} G)$, the bundle of oriented p planes containing $\operatorname{im} G$. $\mathcal{G}_0^p(G)$ is a two sheet covering of N and will be referred to as *the orientation covering of N by G.* $\pi_0: F(G) \to \mathcal{G}_0^p(G)$ is an $SO(p) \times O(m-p)$ fibration, $\pi_1 = z \circ \pi_0: F(G) \to N$ is an $O(p) \times O(m-p)$ fibration. Let

$$\hat{G}|_{(x,b)}(u) = (b|_{R^p})^{-1} \circ G|_{N_x} \circ T\pi_1(u) \tag{3.1.2}$$

define a 1 form $\hat{G}$ on $F(G)$ with values in R^p. $\hat{G}$ is a smooth, π_1-horizontal, ρ_p-equivariant 1 form. Let $G' = (\hat{G})^\#$ be the associated 1 form on N with values in $(\bar{G})^*\mathcal{G}^T$, the vector bundle associated to $F(G)$ by the representation ρ_p. $\hat{G}$ is also π_0-horizontal and $\rho_p|_{SO(p)\times O(m-p)}$ equivariant; let $G'' = (\hat{G})^\#$ be the associated 1 form on $\mathcal{G}_0^p(G)$ with values in $(\bar{G}_*)^*\mathcal{G}_0^T$, the vector bundle associated to $F(G)$ by the representation $\rho_p|_{SO(p)\times O(m-p)}$. Clearly,

$$\delta z G' = G''. \tag{3.1.3}$$

Let $u \in N_x$ and choose $b = (x, b) \in F(G)(x)$ and $\bar{u} \in F(G)_b$ so that $T\pi_1\bar{u} = u$, then by (3.1.2) and (2.2.2a):

$$\begin{aligned} G'(u) &= (x, b)\cdot(\hat{G}(\bar{u})) = (x, b)\cdot(b|_{R^p}^{-1} \circ G|_{N_x} \circ T\pi_1(\bar{u})) \\ &= (x, G|_{N_x}(u)) = g_*^{-1}(G(u)) \end{aligned} \tag{3.1.4}$$

where $g_*: g^*T(M) = \bar{G}^*(\mathcal{G}^T) \oplus \bar{G}^*(\mathcal{G}^\perp) \to T(M)$.

$$G^\perp \equiv \bar{G}^*(\mathcal{G}^\perp) = \{(x, v) \in g^*T(M) | v \in \perp G(N_x) \subset M_{g(x)}\} \tag{3.1.5}$$

is *the normal bundle of* G. The 1 form Φ on $\mathcal{G}^p(M)$ pulls back to a 1 form on N; $\delta\bar{G}\Phi: T(N) \to \bar{G}^* \operatorname{Hom}(\mathcal{G}^T, \mathcal{G}^\perp) = \operatorname{Hom}(\bar{G}^*\mathcal{G}^T, G^\perp)$.

DEFINITION. *The second fundamental form of a TBI G along g* is the N vector bundle map

$$\mathrm{II}_G: G^\perp \to \operatorname{Hom}(T(N), T(N))$$

given by

$$\langle \mathrm{II}_G v(w), w' \rangle_x = -\langle \delta\bar{G}\Phi(w) G'(w'), v \rangle_{g(x)}$$

for $w, w' \in N_x$ and $(x, v) \in G^\perp(x)$.

DEFINITION. The cross section of $G^\perp$ dual with respect to the inner product to $(\text{trace}) \circ \mathrm{II}_G: G^\perp \to N \times R$ is the *mean curvature vector field of* G and is denoted M.C. (G):

$$\langle \text{M.C.}\,(G)(x), v \rangle_{g(x)} = \operatorname{trace}(\mathrm{II}_G(v))$$

for $(x, v) \in G^\perp(x)$.

DEFINITION. G is a *parallel TBI along g* if M.C. $(G) \equiv 0$.

Let g be an isometric immersion and put $G = Tg$. Then $F(G)$ is the usual bundle of adapted frames of g and II_g is the second fundamental form of g, see for example [2, pp. 186–190]. Furthermore, M.C. (G) is the usual mean curvature vector field of g [9, p. 68]. In particular g is a minimal immersion (see [9, p. 71] or [2, p. 248]) if and only if G is parallel along g.

3.2 *The classical case.* The sense in which the present use of the word parallel extends standard usage is made clear in the following proposition. Let $\gamma: (\alpha, \beta) \to M$ be an immersion of an interval (α, β) in a Riemannian manifold M. There is a natural bijective correspondence between the set of TBI's along γ and the set of unit vector fields along γ. In fact, consider (α, β) with its usual Riemannian structure in which $\partial/\partial t$ is the canonical unit cross section of $T((\alpha, \beta))$. For each unit vector field X along γ define a TBI G_X along γ by the formula $G_X(\partial/\partial t(t)) = X(t)$. Each TBI G along γ is uniquely of the form G_X where $X(t) = G(\partial/\partial t(t))$. The correspondence $X \leftrightarrow G_X$ is the desired bijection.

PROPOSITION 3.2.1. *X is a parallel unit vector field along γ if and only if G_X is a parallel TBI along γ.*

Proof. Given G_X let $\bar{G}_X: (\alpha, \beta) \to \mathscr{G}^1(M)$ be the lift of γ defined in §3.1; $\bar{G}_X(t) = \text{span}\,(X(t))$. Choose any $a \in (\alpha, \beta)$ and $b \in \pi_1^{-1}(\bar{G}_X(a))$ so that $b = (X(a), e_2, \ldots, e_m)$. Let $\tilde{\gamma}$ be the horizontal lift of $\bar{G}_X$ through b relative to the connection $\varphi_1 \oplus \varphi_{m-1}$ on $F(M)$ as a bundle over $\mathscr{G}^1(M)$ (see §2.3); $\tilde{\gamma}(t) = (X(t), e_2(t), \ldots, e_m(t))$.

It follows from the definition that G_X is a parallel TBI if and only if $\bar{G}_X$ is a horizontal curve relative to the horizontal distribution $K = \ker \Phi$ on $\mathscr{G}^1(M)$ (see §2.3). But $\bar{G}_X$ is horizontal if and only if $\tilde{\gamma}$ is a horizontal lift of γ relative to the Riemannian connection $\varphi = \varphi_1 \oplus \varphi_{m-1} \oplus \varphi_{1,m-1}$ on $F(M)$. Finally, $\tilde{\gamma}$ is a horizontal lift of γ if and only if X is a parallel unit vector field along γ. Q.E.D.

3.3 $\mu(G)$, *the differential form for parallel TBI*'s. The condition for parallelism of a TBI G along $g: N \to M$ can be restated in terms of a vector bundle valued differential form.

If $p = \text{dimension } N = 1$, $(\bar{G}_*)^*c$ is a nonvanishing section of $(\bar{G}_*)^*\mathscr{G}_0^T$ (see §1.1 and the end of §2.2).

If $p = \text{dimension } N > 1$, then (see the end of §2.2) $(\bar{G}_*)^*h \circ \Lambda^{p-1}(G'')$ is a $p-1$ form on $\mathscr{G}_0^p(G)$ with values in $(\bar{G}_*)^*\mathscr{G}_0^T$.

DEFINITION. If $p = 1$, the *mean curvature form* of G is the differential one form $\mu(G)$ on $\mathscr{G}_0^1(G)$ with values in the normal bundle $G^\perp$, given by

$$\mu(G) = \delta\bar{G}_*\Phi_0(\ )[(\bar{G}_*)^*c].$$

If $p > 1$, the *mean curvature form* of G is the differential p form $\mu(G)$ on $\mathscr{G}_0^p(G)$ with values in the normal bundle $G^\perp$, given by

$$\mu(G) = [\delta\bar{G}_*\Phi_0] \wedge_{(\bar{G}_*)^*S} [(\bar{G}_*)^*h \circ \Lambda^{p-1}(G'')]$$

where S is the natural bilinear pairing

$$\text{Hom}\,(\mathscr{G}_0^T, \mathscr{G}_0^\perp) \oplus \mathscr{G}_0^T \to \mathscr{G}_0^\perp$$

(see §1.1 and (1.2.1)).

PROPOSITION 3.3.1. *A TBI G along g is parallel if and only if the mean curvature form $\mu(G)$ vanishes identically on the orientation covering $\mathscr{G}_0^p(G)$.*

Proof. Let $p=1$. Since $Tz\colon T\mathscr{G}_0^1(G)\to T(N)$ is surjective, $\delta\bar{G}\Phi$ vanishes if and only if $\delta z\delta\bar{G}\Phi=\delta\bar{G}_*\delta z\Phi=\delta\bar{G}_*\Phi_0$ vanishes. But the vanishing of $\delta\bar{G}\Phi$ is equivalent to parallelism.

Suppose now that $p>1$ and that $x\in N$. Let $P^+=(x, G(N_x)^+)\in\mathscr{G}_0^p(G)$. Choose a basis $v_1,\dots,v_p$ of $\mathscr{G}_0^p(G)_{P^+}$ so that $Tz(v_1),\dots,Tz(v_p)$ is an orthonormal basis of N_x and

$$(*)\qquad G''(v_1)\wedge\cdots\wedge G''(v_p) = + = (\bar{G}_*)^*c(P^+)\in\Lambda^p(\bar{G}_*)^*\mathscr{G}_0^T(P^+).$$

By the fact that $\delta z\Phi=\Phi_0$ and (3.1.1) it follows that $\delta(z\circ\bar{G})\Phi=\delta\bar{G}_*\Phi_0$. Thus

$$\begin{aligned}
z_*^{-1}(\text{M.C. }(G)(x)) &= -z_*^{-1}\sum_{i=1}^{p}\delta\bar{G}\Phi(Tz(v_i))[G'(Tz(v_i))]\\
&= -\sum_{i=1}^{p} z_*^{-1}\delta\bar{G}\Phi(Tz(v_i))[z_*^{-1}G'(Tz(v_i))]\\
&= -\sum_{i=1}^{p}\delta\bar{G}_*\Phi_0(v_i)[G''(v_i)] \qquad \text{by (2.2.4) and (1.2.1)},\\
&= \frac{(-1)^p}{(p-1)!}\sum_{i=1}^{p}(-1)^{i+1}\delta\bar{G}_*\Phi_0(v_i)[(\bar{G}_*)^*h\circ\Lambda^{p-1}G''(v_1,\dots,\hat{v}_i,\dots,v_p)]\\
&= \frac{(-1)^p}{(p-1)!}\mu(G)(v_1,\dots,v_p). \qquad \text{Q.E.D.}
\end{aligned}$$

For the remainder of this paper proofs will be carried out only for the case $p>1$. Nonetheless, the case $p=1$, which corresponds to the classical theorems, can be proved by the methods that follow using the mean curvature form with $p=1$ in the definition above.

4. Minimal embeddings.

4.1 $(\omega_0^\perp, \mu)$, *the differential forms for a minimal immersion.* Define a differential p form μ on $\mathscr{G}_0^p(M)$ with values in $\mathscr{G}_0^\perp$ by the formula

$$\mu = \Phi_0\wedge_S(h\circ\Lambda^{p-1}\omega_0^T).$$

For the definitions of Φ_0, ω_0^T, $\omega_0^\perp$ see §2.3. For S see §3.3.

PROPOSITION 4.1.1. (a) *Let $g\colon N^p\to M^m$ be a minimal immersion. Put $G=Tg$. Then $(\mathscr{G}_0^p(G), \bar{G}_*)$ is an integral manifold of the pair $(\omega_0^\perp, \mu)$ on $\mathscr{G}_0^p(M)$.* (*See diagram* (3.1.1).)

(b) *Conversely, suppose that $(\bar{N}^p$, inclusion$)$ is a p-dimensional integral manifold of $(\omega_0^\perp, \mu)$ which is A_0 invariant* (see §2.2) *and intersects each p_0-fiber of $\mathscr{G}_0^p(M)$ in at most two points. Then $p_0(\bar{N}^p)=N^p$ is a submanifold of M, $(N^p$, inclusion$)$ is a minimal embedding and $\bar{N}^p$ is the image of the orientation covering of N^p by $G=T($inclusion$)$ under the map $\bar{G}_*$.*

Proof. (a) For any isometric immersion g, with $G=Tg$, $(\mathscr{G}_0^p(G), \bar{G}_*)$ is an integral manifold of $\omega_0^\perp$ because $p_0\circ\bar{G}_*=p_1\circ z\circ\bar{G}_*=g\circ z$ in diagram (3.1.1) (see (2.3.5b)).

For any isometric immersion g, with $G=Tg$, $\delta\bar{G}_*\mu=\mu(G)$. In fact, it suffices to show that $\delta\bar{G}_*\omega_0^T=G''$. In (3.1.1), $F(G)=\bar{G}^*F(M)=(\bar{G}_*)^*F(M)$ so $\pi_2\circ(\bar{G}_*)_*=g\circ\pi_1\colon F(G)\to M$. Thus

$$\delta((\bar{G}_*)_*)\omega_p = (b|_{R^p})^{-1}\circ T\pi_2\circ T((\bar{G}_*)_*) = (b|_{R^p})^{-1}\circ G\circ T\pi_1$$

so that $\delta((\bar{G}_*)_*)\omega_p=\hat{G}$, by (3.1.2). Since $(\hat{G})^\#=G''$ and $\omega_p{}^\#=\omega_0^T$ it follows from (1.2.5) that

$$\delta\bar{G}_*\omega_0^T = [\delta((\bar{G}_*)_*)\omega_p]^\# = G''.$$

Hence $\delta\bar{G}_*\mu=\mu(G)$. Part (a) then follows from Proposition 3.3.1.

(b) Since $\bar{N}$ is transversal to fibers and A_0 invariant, N is a submanifold of M. Consider the inclusion $i\colon N\to M$ as an isometric embedding so that $G=Ti$ is a TBI along i. Since $\bar{N}$ is a p-dimensional A_0-invariant integral manifold of $\omega_0^\perp$ that intersects the fibers of $p_0^{-1}(N)$ in exactly two points, $\bar{N}$ is the image of the orientation covering of N by G under the map $\bar{G}_*$. For if $P^+\in\bar{N}$, $N_{p_0(P^+)}=P$. As in (a) $\mu(G)=\delta\bar{G}_*\mu$. Since $\bar{N}$ is an integral manifold of μ, G is a parallel TBI along i by Proposition 3.3.1. Hence (N, i) is a minimal embedding. Q.E.D.

4.2 *The regular integral planes of* $(\omega_0^\perp, \mu)$. Let $P^+\in\mathscr{G}_0^p(M)$ then $I(\omega_0^\perp, P^+)$ is an ideal in the Grassmann algebra $\Lambda(\mathscr{G}_0^p(M)_{P^+})^*$ (see §1.3).

LEMMA 4.2.1. *$I(\omega_0^\perp, P^+)$ is generated as an ideal by the space of one forms*

$$(\omega_0^\perp)^t(\mathscr{G}_0^\perp(P^+))^* \subset \Lambda^1(\mathscr{G}_0^p(M)_{P^+})^*$$

and the space of two forms

$$(\Phi_0\wedge_S\omega_0^T)^t(\mathscr{G}_0^\perp(P^+))^* \subset \Lambda^2(\mathscr{G}_0^p(M)_{P^+})^*$$

(*where t denotes transpose*).

Proof. Choose a neighborhood $\mathscr{U}$ of $P^+\in\mathscr{G}_0^p(M)$ and a section $\chi\colon\mathscr{U}\to F(M)$ such that

$$(*)\qquad T\chi(\mathscr{G}_0^p(M)_{P^+}) = \bar{H}(\chi(P^+)) = H(\chi(P^+))+\operatorname{Ker}(\varphi_p\oplus\varphi_{m-p})(\chi(P^+))$$

(see §2.3). By the identification (1.2.3), $\pi_0^*\mathscr{G}_0^\perp=F(M)\times R^{m-p}$ so that χ determines a local vector chart for $\mathscr{G}_0^\perp$ over $\mathscr{U}$:

$$\mathscr{G}_0^\perp|_{\mathscr{U}} = \chi^*\pi_0^*\mathscr{G}_0^\perp = \chi^*(F(M)\times R^{m-p}) = \mathscr{U}\times R^{m-p}.$$

In fact, if $\bar{r}_1,\dots,\bar{r}_{m-p}$ are the canonical sections of $F(M)\times R^{m-p}$, then $\chi^*r_1,\dots,\chi^*r_{m-p}$ span $\mathscr{G}_0^\perp$ over $\mathscr{U}$ and $\chi^*\bar{r}_i(Q^+)=\chi(Q^+)\cdot r_i$ for all $Q^+\in\mathscr{U}$. By equation (1.2.4),

$$\delta\pi_0\omega_0^T = \omega_p,\qquad \delta\pi_0\omega_0^\perp = \omega_{m-p}\quad\text{and}\quad \delta\pi_0\Phi_0 = \varphi_{p,m-p},$$

so that

$$\omega_0^T|_{\mathscr{U}} = \delta\chi\delta\pi_0\omega_0^T = \delta\chi\omega_p,\qquad \omega_0^\perp|_{\mathscr{U}} = \delta\chi\omega_{m-p}\quad\text{and}\quad \Phi_0|_{\mathscr{U}} = \delta\chi\varphi_{p,m-p}.$$

There are 1 forms ω^i_{m-p} on $F(M)$ and 1 forms $\omega_0^{\perp i}$ on $\mathscr{U}$ such that

$$\omega_{m-p} = \sum_{i=1}^{m-p} \omega^i_{m-p}\bar{r}_i \quad \text{and} \quad \omega_0^{\perp}|_{\mathscr{U}} = \sum_{i=1}^{m-p} \omega_0^{\perp i}\chi^*\bar{r}_i.$$

So

$$\sum_{i=1}^{m-p} \omega_0^i\chi^*\bar{r}_i = \omega_0^{\perp}|_{\mathscr{U}} = \delta\chi\omega_{m-p} = \sum_{i=1}^{m-p} \delta\chi(\omega^i_{m-p})\chi^*\bar{r}_i.$$

Hence $I(\omega_0^{\perp}, P^+)$ is generated as an ideal by the 1 forms

$$\omega_0^{\perp i} = \delta\chi(\omega^i_{m-p}), \qquad i = 1,\ldots, m-p,$$

and their differentials

$$d(\omega_0^{\perp i}) = d\delta\chi(\omega^i_{m-p}) = \delta\chi d(\omega^i_{m-p}), \qquad i = 1,\ldots, m-p.$$

By (2.3.3b),

$$-\varphi_{p,m-p} \wedge \omega_p - \varphi_{m-p,m-p} \wedge \omega_{m-p} = d\omega_{m-p} = \sum_{i=1}^{m-p} (d\omega^i_{m-p})\bar{r}_i$$
$$= \sum_{i=1}^{m-p} d(\omega^i_{m-p})\bar{r}_i.$$

By $(*)$, at P^+

$$-\sum_{i=1}^{m-p} (\Phi_0 \wedge \omega_0^T)^i\chi^*\bar{r}_i \equiv -\Phi_0 \wedge \omega_0^T = -\delta\chi(\varphi_{p,m-p} \wedge \omega_p)$$
$$= \delta\chi d\omega_{m-p} = \sum_{i=1}^{m-p} \delta\chi d(\omega^i_{m-p})\chi^*\bar{r}_i.$$

Hence $d(\omega_0^{\perp i}) = -(\Phi_0 \wedge \omega_0^T)^i$, $i=1,\ldots, m-p$. Q.E.D.

Suppose that $E^q \subset \mathscr{G}_0^p(M)_{P^+}$, $0 \leqq q \leqq p-1$, is an integral q plane of $\omega_0^{\perp}$ which is complementary to the vertical. Define

$$Q \equiv Tp_0E^q \subset P \subset M_x \quad \text{and} \quad U \equiv P \cap \perp Q.$$

Under the isomorphism, Proposition 2.3.1, Φ_0: $V^{p_0}(P^+) \to \operatorname{Hom}(P, \perp P)$, $V^{p_0}(P^+)$ decomposes into a direct sum of the subspaces

$$\alpha \equiv \Phi_0^{-1} \operatorname{Hom}(U; P, \perp P) \quad \text{and} \quad \beta \equiv \Phi_0^{-1} \operatorname{Hom}(Q; P, \perp P).$$

Let

$$\gamma \equiv (Tp_0)^{-1}U \cap K_0(P^+) \qquad \text{(see §2.3).}$$

There is a direct sum decomposition

$$(Tp_0)^{-1}P = E^q \oplus \gamma \oplus \alpha \oplus \beta.$$

LEMMA 4.2.2. (a) *With* $E^q \subset \mathscr{G}_0^p(M)_{P^+}$ *as above,*

$$H(E^q, \omega_0^{\perp}) = E^q \oplus \operatorname{Graph} L_\alpha \oplus \beta \subset (Tp_0)^{-1}P$$

where $L_\alpha\colon \gamma \to \alpha$ *is the unique linear transformation such that*

$$\Phi_0 \circ L_\alpha(v)(\omega_0^T(u)) = \Phi_0(u)(\omega_0^T(v))$$

for every $v \in \gamma$ *and* $u \in E^q$. *In particular,* $\dim H(E^q, \omega_0^\perp) = p+(p-q)(m-p)$ *and* E^q *is a regular integral plane of* $\omega_0^\perp$.

(b) *Put* $q=p-1$ *in part* (a). E^{p-1} *is an integral plane of* μ. *Let* $\{v_1, \dots, v_{p-1}\}$ *be a basis of* E^{p-1} *and* $\{v_p\}$ *a basis of* γ *such that* $Tp_0v_1, \dots, Tp_0v_p$ *is an orthonormal basis of* P *and* $Tp_0v_1 \wedge \cdots \wedge Tp_0v_p = +$. *Then*

$$H(E^{p-1}, (\omega_0^\perp, \mu)) = E^{p-1} \oplus \mathrm{Graph}\,(L_\alpha \oplus L_\beta)$$

where $L_\alpha \oplus L_\beta\colon \gamma \to \alpha \oplus \beta$, L_α *is as above and* $L_\beta\colon \gamma \to \beta$ *is the unique linear transformation such that*

$$\Phi \circ L_\beta(v_p)(\omega_0^T(v_p)) = -\sum_{i=1}^{p-1} \Phi_0(v_i)(\omega_0^T(v_i)).$$

In particular, $H(E^{p-1}, (\omega_0^\perp, \mu))$ *is a p-dimensional complement to the vertical and* E^{p-1} *is a regular integral plane of* $(\omega_0^\perp, \mu)$.

Proof. (a) Each $w \in (Tp_0)^{-1}P$ may be written uniquely as

$$w = e+v+a+b \in E^q \oplus \gamma \oplus \alpha \oplus \beta.$$

By the preceding lemma, $w \in H(E^q, \omega_0^\perp)$ if and only if $w \in \operatorname{Ker} w_0^\perp = (Tp_0)^{-1}P$ and $w \in \operatorname{Ker} i(u)(\Phi_0 \wedge_S \omega_0^T)$ for each $u \in E^q$. Thus $w \in H(E^q, \omega_0^\perp)$ if and only if, for each $u \in E^q$,

$$0 = \Phi_0 \wedge_S \omega_0^T(u, w) = \Phi_0 \wedge_S \omega_0^T(u, e+v+a+b).$$

Since E^q is an integral plane, v is horizontal, $a+b$ is vertical and $\omega_0^T(u) \in Q$, the last statement is equivalent to $\Phi(u)(\omega_0^T(v)) = \Phi_0(a)(\omega_0^T(u))$ for each $u \in E^q$ or $a = L_\alpha(v)$. The regularity of the integral plane E^q follows from the above using induction and the fact that integral planes complementary to the vertical are open in the set of integral planes.

(b) $H(E^{p-1}, (\omega_0^\perp, \mu)) = H(E^{p-1}, \omega_0^\perp) \cap \operatorname{Ker} i(v_1 \wedge \cdots \wedge v_{p-1})\mu$. Thus,

$$w \in H(E^{p-1}, (\omega_0^\perp, \mu))$$

if and only if

$$w = e+cv_p+L_\alpha(cv_p)+b \in E^{p-1} \oplus \mathrm{Graph}\, L_\alpha \oplus \beta$$

and

$$0 = i(v_1 \wedge \cdots \wedge v_{p-1})\mu(w).$$

Equivalently,

$$\begin{aligned} 0 &= \Phi_0 \wedge_S h \circ \Lambda^{p-1}\omega_0^T(v_1, \dots, v_{p-1}, cv_p+L_\alpha(cv_p)+b) \\ &= \sum_{i=1}^{p-1} (-1)^{i+1}\Phi_0(v_i)[h \circ \Lambda^{p-1}\omega_0^T(v_1, \dots, \hat{v}_i, \dots, v_{p-1}, cv_p+L_\alpha(cv_p)+b)] \\ &\qquad + (-1)^{p+1}\Phi_0(cv_p+L_\alpha(cv_p)+b)(h \circ \Lambda^{p-1}\omega_0^T(v_1, \dots, v_{p-1})) \\ &= (-1)^{p+1}(p-1)!\left[c\sum_{i=1}^{p-1} \Phi_0(v_i)(\omega_0^T(v_i)) + \Phi_0(b)(\omega_0^T(v_p))\right] \end{aligned}$$

since $\omega_0^T(v_p) \in \operatorname{Ker} \Phi_0 \circ L_\alpha(v_p)$. Equivalently,

$$-c \sum_{i=1}^{p-1} \Phi_0(v_i)(\omega_0^T(v_i)) = \Phi_0(b)(\omega_0^T(v_p))$$

or $b = L_\beta(v_p)$. Regularity of E^{p-1} follows as in part (a). Q.E.D.

4.3 *The local existence and uniqueness of minimal embeddings.*

LEMMA 4.3.1. *If $E^p \subset \mathscr{G}_0^p(M)_{P^+}$ is an integral p plane of $(\omega_0^\perp, \mu)$ complementary to the vertical, then so is $TA_0E^p \subset \mathscr{G}_0^p(M)_{P^-}$.*

Proof. Let $v_1, \ldots, v_p$ be a basis of E^p such that $Tp_0(v_1), \ldots, Tp_0(v_p)$ is an orthonormal basis of $P \subset M_{p_0(P)}$ with

$$Tp_0(v_1) \wedge \cdots \wedge Tp_0(v_p) = +.$$

Choose $b \in F(M)(P^+)$ and $\bar{v}_1, \ldots, \bar{v}_p \in F(M)_b$ such that $T\pi_0(\bar{v}_i) = v_i$, $i = 1, \ldots, p$. Let $\sigma \in O(p) \times O(m-p) \subset O(m)$ where $\sigma_{ij} = (-1)^{\delta(i,p)} \cdot \delta(i,j)$ [$\delta(i,j)$ is the Kronecker Delta Function]. Then $b\sigma \in F(M)(P^-)$, $\pi_0 \circ r_\sigma = A_0 \circ \pi_0$ so that $\{T\pi_0(Tr_\sigma \bar{v}_i) = TA_0(v_i), i = 1, \ldots, p\}$ span $TA(E^p)$ and $p_0 = p_0 \circ A_0$ so that

$$Tp_0(TA_0(v_1)) \wedge \cdots \wedge Tp_0(TA_0(v_{p-1})) \wedge Tp_0(TA_0(-v_p)) = -.$$

Since $\varphi_{p,m-p}$ is Hom (ρ_p, ρ_{m-p}) equivariant and ω_p is ρ_p-equivariant, it follows that for each i and j,

$$\begin{aligned}\varphi_{p,m-p} \circ Tr_\sigma(\bar{v}_i)[\omega_p(Tr_\sigma \bar{v}_j)] &= \delta r_\sigma \varphi_{p,m-p}(\bar{v}_i)[\delta r_\sigma \omega_p(\bar{v}_j)] \\ &= \sigma^{-1}|_{R^{m-p}} \circ \varphi_{p,m-p}(\bar{v}_i) \circ \sigma|_{R^p}[\sigma^{-1}|_{R^p} \circ \omega_p(\bar{v}_j)] \\ &= \varphi_{p,m-p}(\bar{v}_i)[\omega_p(\bar{v}_j)].\end{aligned}$$

By Lemma 4.2.1, the $(\omega_0^\perp, \mu)$-integrability of E^p and $TA_0(E^p)$ is equivalent to the vanishing of $\omega_0^\perp$, $\Phi_0 \wedge_S \omega_0^T$ and μ on these planes.

Since (equation (2.3.5b)) $\omega_0^\perp|_{P^+} =$ (projection on P) $\circ\, Tp_0 = \omega_0^\perp|_{P^-}$ and $p_0 = p_0 \circ A_0$, the vanishing of $\omega_0^\perp$ on E^p implies that it also vanishes on TA_0E^p.

Since, on E^p, for all i and j,

$$\begin{aligned}0 = \Phi_0 \wedge_S \omega_0^T(v_i, v_j) &= \Phi_0(v_i)[\omega_0^T(v_j)] - \Phi_0(v_j)[\omega_0^T(v_i)] \\ &= b \cdot [\varphi_{p,m-p}(\bar{v}_i)(\omega_p(\bar{v}_j)) - \varphi_{p,m-p}(\bar{v}_j)(\omega_p(\bar{v}_i))],\end{aligned}$$

it follows that on $TA_0(E^p)$, for all i and j,

$$\begin{aligned}&\Phi_0 \wedge_S \omega_0^T(TA_0v_i, TA_0v_j) \\ &\quad = b \cdot [\varphi_{p,m-p}(Tr_\sigma \bar{v}_i)(\omega_p(Tr_\sigma \bar{v}_j)) - \varphi_{p,m-p}(Tr_\sigma \bar{v}_j)(\omega_p(Tr_\sigma \bar{v}_i))] = 0,\end{aligned}$$

so that $TA_0(E^p)$ is an integral plane of $\omega_0^\perp$.

Since, on E^p,

$$\begin{aligned}0 = \mu(v_1, \ldots, v_p) &= (-1)^{p+1}(p-1)! \sum_{i=1}^{p} \Phi_0(v_i)[\omega_0^T(v_i)] \\ &= (-1)^{p+1}(p-1)!\, b \cdot \left[\sum_{i=1}^{p} \varphi_{p,m-p}(\bar{v}_i)(\omega_p(\bar{v}_i))\right],\end{aligned}$$

it follows that on $TA_0(E^p)$,

$$\begin{aligned}\mu(TA_0(v_1), \ldots, TA_0(v_{p-1}), TA_0(-v_p)) &= (-1)^{p+1}(p-1)!\, b\sigma\cdot\left[\sum_{i=1}^{p} \varphi_{p,m-p}(Tr_\sigma\bar{v}_i)(\omega_p(Tr_\sigma\bar{v}_i))\right]\\ &= 0,\end{aligned}$$

so that $TA_0(E^p)$ is an integral plane of $(\omega_0^\perp, \mu)$. Q.E.D.

THEOREM A (p). 4.3.2. *See* [9, *pp.* 71–72]. *Let* (N^{p-1}, i) *be a homeomorphically embedded submanifold of the Riemannian manifold* M^m. *Let* $D\colon N^{p-1} \to \mathscr{G}^p(M)$ *be a p-dimensional distribution along the inclusion i that contains the tangent planes of* N^{p-1}, *i.e.* $Ti(N_x^{p-1}) \subset D(x)$ *for all* $x \in N^{p-1}$.

Then assuming that the data is real analytic, for each $x_0 \in N^{p-1}$ *there is a neighborhood* $\mathscr{U}$ *of* x_0 *in M and a unique minimally embedded p-dimensional real analytic submanifold* N^p *of M that extends the initial data*:

(1) $C_{x_0}(N^{p-1} \cap \mathscr{U}) \subset N^p \subset \mathscr{U}$,

(2) $N_x^p = D(x)$ *for all* $x \in C_{x_0}(N^{p-1} \cap \mathscr{U})$,

where $C_{x_0}(N^{p-1} \cap \mathscr{U})$ *is the connected component of* x_0 *in* $N^{p-1} \cap \mathscr{U}$.

Proof. $D^*\mathscr{G}_0^p(M)$ is a two fold cover of N^{p-1} and

$$\bar{N}^{p-1} \equiv D_*(D^*\mathscr{G}_0^p(M))$$

is contained in $\mathscr{G}_0^p(M)$ as a $(p-1)$-dimensional integral manifold of $(\omega_0^\perp, \mu)$ that is transversal to fibers. Given $x_0 \in N^{p-1}$, fix $P^+ = D(x_0)^+ \in \bar{N}^{p-1}$ and let $E^{p-1} = (\bar{N}^{p-1})_{P^+}$. By Lemma 4.2.2(b), E_+^{p-1} is a regular integral plane of $(\omega_0^\perp, \mu)$ and $H(E^{p-1}, (\omega_0^\perp, \mu))$ is a p-dimensional complement to the vertical. By the Cartan-Kahler Theorem 1.4.1 there is a neighborhood $\mathscr{U}_+$ of P^+ in $\mathscr{G}_0^p(M)$ and a unique p-dimensional integral manifold $\bar{N}_+^p$ of $(\omega_0^\perp, \mu)$ passing through P^+, contained in $\mathscr{U}_+$ and containing the connected component of $\bar{N}^{p-1} \cap \mathscr{U}_+$ through P^+. Assume that $\mathscr{U}_+$ is sufficiently small so that $\bar{N}_+^p$ projects, under p_0, diffeomorphically onto its image; thus $\bar{N}_+^p \cap A_0(\bar{N}_+^p)$ is empty. Let $P^- = A_0P^+$, $E_-^{p-1} = TA_0E_+^{p-1}$, $\mathscr{U}_- = A_0(\mathscr{U}_+)$ and $\bar{N}_-^p = A_0(\bar{N}_+^p)$. By Lemma 4.3.1, $\bar{N}_-^p$ is an integral manifold of $(\omega_0^\perp, \mu)$. By the uniqueness in Cartan-Kahler (at E_-^{p-1}), $\bar{N}_-^p$ is the unique p-dimensional integral manifold of $(\omega_0^\perp, \mu)$ passing through P^-, contained in $\mathscr{U}_-$ and containing the connected component of $\bar{N}^{p-1} \cap \mathscr{U}_-$ through P^-. Let $\bar{N}^p = \bar{N}_+^p \cup \bar{N}_-^p$. By Proposition 4.1.1 $N^p = p_0(\bar{N}^p)$ is a minimally embedded submanifold of M. N^p extends the initial data on $\mathscr{U} = p_0(\mathscr{U}_+)$. Uniqueness follows from the Cartan-Kahler Theorem again. Q.E.D.

5. The existence and uniqueness of parallel TBI's.

5.1 *The bundle of partial data.* In order to investigate the existence and uniqueness of parallel TBI's along $g\colon N^p \to M^m$ it will be necessary to deal with two types of initial data. One type is called *partial data.* To be precise let D be a smooth

$(p-1)$-dimensional distribution on N^p and let D also denote the $(p-1)$-dimensional subbundle of $T(N)$ that it determines. Let $G^{p-1}\colon D \to T(M)$ be a smooth Euclidean map along g. The pair (G^{p-1}, D) is the partial data. Given (G^{p-1}, D), the problem is to determine those parallel TBI G along g such that $G|_D = G^{p-1}$. According to Theorem B (p) of §5.4, these extensions of (G^{p-1}, D) are parametrized locally by a second type of initial data given along a codimension one submanifold of N^p.

Given (G^{p-1}, D) let

$$F(G^{p-1}) \equiv F(\text{im } G^{p-1}) \subset g^*F(M)$$

as in §2.3 where im G^{p-1} is the distribution along g:

$$\text{im } G^{p-1}(x) = G^{p-1}(D(x)) \subset M_{g(x)}$$

for all $x \in N^p$. $i\colon F(G^{p-1}) \to g^*F(M)$ is a reduction of the group of $g^*F(M)$ as a bundle over N to the subgroup $O(p-1)\times O(m-p+1)$. Similarly, $\mathscr{G}_0^p(G^{p-1}) \equiv \mathscr{G}_0^p(\text{im } G^{p-1})$ is included, $i\colon \mathscr{G}_0^p(G^{p-1}) \to g^*\mathscr{G}_0^p(M)$ as a subbundle over N. In addition, $i(F(G^{p-1}))$ is contained in $g^*F(M)|\mathscr{G}_0^p(G^{p-1}) = i^*g^*F(M)$ and reduces the group $SO(p)\times O(m-p)$ of $i^*g^*F(M)$ as a bundle over $\mathscr{G}_0^p(G^{p-1})$ to the subgroup $S\Gamma(p-1, p)$. $\mathscr{G}^p(G^{p-1}) \equiv \mathscr{G}^p(\text{im } G^{p-1})$ has similar properties.

To each $(x, P^+) \in \mathscr{G}_0^p(G^{p-1})$ there corresponds exactly two linear isometries $N_x \to P$ which extend $G^{p-1}\colon D(x) \to M_{g(x)}$. The set $E\mathscr{G}_0^p(G^{p-1})$ of all such extensions for all $(x, P^+) \in \mathscr{G}_0^p(G^{p-1})$ is in a natural way an $O(1)$ bundle over (hence, a two fold covering of) $\mathscr{G}_0^p(G^{p-1})$. $E\mathscr{G}_0^p(G^{p-1})$ is the *bundle of* (the extensions of the) *partial data*. The bundle projection $\eta\colon E\mathscr{G}_0^p(G^{p-1}) \to \mathscr{G}_0^p(G^{p-1})$ is the map $(x, P^+$, extension of $G^{p-1}|_{D(x)}$ to $N_x \to P) \mapsto (x, P^+)$. The bundle structure is obtained in the following manner. For any $(x, P^+) \in \mathscr{G}_0^p(G^{p-1})$, choose a neighborhood $\mathscr{U}$ of (x, P^+) on which there is both a smooth section $\chi\colon \mathscr{U} \to F(G^{p-1})$ $(\chi(\bar{x}, \bar{P}^+) = [\bar{x}, \bar{P}^+, e_1(\bar{x}, \bar{P}^+), \ldots, e_m(\bar{x}, \bar{P}^+)])$ and a smooth unit vector field Y on $p_0(\mathscr{U}) \subset N^p$, such that Y is orthogonal to the distribution D. Define

$$\psi_{\mathscr{U}}\colon O(1)\times \mathscr{U} \to \eta^{-1}(\mathscr{U})$$

by $\psi_{\mathscr{U}}(1, \bar{x}, \bar{P}^+) = (\bar{x}, \bar{P}^+, G_+)$ and $\psi_{\mathscr{U}}(-1, \bar{x}, \bar{P}^+) = (\bar{x}, \bar{P}^+, G_-)$ where G_+ and G_- are the two extensions of $G^{p-1}\colon D(\bar{x}) \to M_{g(\bar{x})}$ to isometries $N_{\bar{x}} \to \bar{P}$ and $G_+(Y(\bar{x})) = e_p(\bar{x}_1 \bar{P}^+)$ while $G_-(Y(\bar{x})) = -e_p(\bar{x}, \bar{P}^+)$. Declare the bijection $\psi_{\mathscr{U}}$ to be a diffeomorphism and construct a bundle structure using such maps $\psi_{\mathscr{U}}$ as the strip maps.

$\eta^*F(G^{p-1})$ can be identified with $\pi_0^*E\mathscr{G}_0^p(G^{p-1})$: $[(x, P^+, G_+), (x, b)] \leftrightarrow [(x, b), (x, P^+, G_+)]$. $\eta^*F(G^{p-1})$ is thus a $S\Gamma(p-1, p)$ principal bundle over $E\mathscr{G}_0^p(G^{p-1})$ with projection π_0 and an $O(1)$ bundle over $F(G^{p-1})$ with projection η_*.

The preceding discussion is summarized in the following commutative diagram in which the vertical maps are all bundle projections, i is inclusion and I is the identity:

$$\begin{tikzcd}
\eta^*F(G^{p-1}) \arrow[r, "\eta_*"] \arrow[d, "\pi_0"] & F(G^{p-1}) \arrow[r, "i"] \arrow[d, "\pi_0"] \arrow[rr, bend left, "i"] & g^*F(M)|\mathscr{G}^p_0(G^{p-1}) \arrow[r, "i_*"] \arrow[d, "\pi_0"] & g^*F(M) \arrow[r, "g_*"] \arrow[d, "\pi_0"] & F(M) \arrow[d, "\pi_0"] \arrow[dd, bend left, "\pi_2"] \\
E\mathscr{G}^p_0(G^{p-1}) \arrow[r, "\eta"] & \mathscr{G}^p_0(G^{p-1}) \arrow[r, leftrightarrow, "I"] \arrow[d, "p_0"] & \mathscr{G}^p_0(G^{p-1}) \arrow[r, "i"] \arrow[d, "p_0"] & g^*\mathscr{G}^p_0(M) \arrow[r, "g_*"] \arrow[d, "p_0"] & \mathscr{G}^p_0(M) \arrow[d, "p_0"] \\
 & N \arrow[r, leftrightarrow, "I"] & N \arrow[r, leftrightarrow, "I"] & N \arrow[r, "g"] & M
\end{tikzcd} \tag{5.1.1}$$

The notation in this diagram together with the abbreviations

$$k = p_0 \circ \eta \colon E\mathscr{G}^p_0(G^{p-1}) \to N \tag{5.1.2}$$

and

$$s = g_* \circ i \circ \eta \colon E\mathscr{G}^p_0(G^{p-1}) \to \mathscr{G}^p_0(M)$$

will remain in force for partial data (G^{p-1}, D) for the rest of this section.

5.2 $\mu(G^{p-1})$, *the differential form for a parallel extension.* Define a 1 form τ on $\eta^*F(G^{p-1})$ with values in R^p by the formula

$$\tau_{(x,b,G_+)} = (b|_{R^p})^{-1} \circ G_+ \circ T(\pi_2 \circ \eta_*) \tag{5.2.1}$$

for $(x, b, G_+) = [(x, P^+, G_+)(x, b)] \in \eta^*F(G^{p-1})$. A routine computation shows that τ is smooth, it is π_0-horizontal since $\operatorname{Ker} T\pi_0 \subset \operatorname{Ker} T(\pi_2 \circ \eta_*)$. τ is equivariant with respect to $\rho_p|_{S\Gamma(p-1,p)}$ since

$$\begin{aligned}
\rho_p(\sigma^{-1})\tau|_{(x,b,G_+)}(v) &= \sigma^{-1}|_{R^p} \circ (b|_{R^p})^{-1} \circ G_+ \circ T(\pi_2 \circ \eta_*)(v) \\
&= (b\sigma|_{R^p})^{-1} \circ G_+ \circ T(\pi_2 \circ \eta_*) \circ Tr_\sigma(v) \\
&= \tau|_{(x,b\sigma,G_+)} Tr_\sigma(v) = \delta r_\sigma \tau|_{(x,b,G_+)}(v)
\end{aligned}$$

for all $v \in \eta^*F(G^{p-1})_{(x,b,G_+)}$ and $\sigma \in S\Gamma(p-1, p)$. Let $\tau^\#$ be the associated 1 form on $E\mathscr{G}^p_0(G^{p-1})$ with values in $s^*\mathscr{G}^T_0$, the vector bundle associated to $\eta^*F(G^{p-1})$ by the representation $\rho_p|_{S\Gamma(p-1,p)}$. Suppose that $u \in E\mathscr{G}^p_0(G^{p-1})_{(P^+,G_+)}$; choose $(x, b, G_+) \in \eta^*F(G^{p-1})(P^+, G_+)$ and pick $\bar{u} \in \eta^*F(G^{p-1})_{(x,b,G_+)}$ so that $T\pi_0\bar{u} = u$. Then

$$\begin{aligned}
\tau^\#(u) &= (x, b, G_+)\cdot(\tau(\bar{u})) \\
&= (x, b, G_+)\cdot[(b|_{R^p})^{-1} \circ G_+ \circ T(\pi_2 \circ \eta_*)(\bar{u})] \\
&= (x, b, G_+)\cdot[(b|_{R^p})^{-1} \circ G_+ \circ T(p_0 \circ \eta)(u)]
\end{aligned}$$

so that

$$\tau^\#|_{(P^+,G_+)} = G_+ \circ Tk \tag{5.2.2}$$

under the identification $s^*\mathscr{G}^T_0(P^+, G_+) = P$, of §2.2.

DEFINITION. $\mu(G^{p-1})$, *the differential form for a parallel extension of* (G^{p-1}, D) is the p form on $E\mathscr{G}_0^p(G^{p-1})$ with values in $s^*\mathscr{G}_0^{\perp}$ given by the formula

$$\mu(G^{p-1}) = \delta s\Phi_0 \wedge_{s\cdot s} (s^*h \circ \Lambda^{p-1}\tau^{\#}).$$

This terminology is justified in the next proposition.

Suppose that G is a TBI along g that extends (G^{p-1}, D) so that $G|_D = G^{p-1}$. Refer to diagrams (3.1.1) and (5.1.1). The map $\bar{G}_*: \mathscr{G}_0^p(G) \to \mathscr{G}_0^p(M)$ factors into the map $d: \mathscr{G}_0^p(G) \to \mathscr{G}_0^p(G^{p-1})$; $(x, P^+) \to (x, P^+)$ (where $P = G(N_x)$), followed by $g_* \circ i: \mathscr{G}_0^p(G^{p-1}) \to \mathscr{G}_0^p(M)$. d lifts to a map

$$L: \mathscr{G}_0^p(G) \to E\mathscr{G}_0^p(G^{p-1});\ (x, P^+) \mapsto (x, P^+, G|_{N_x})$$

followed by η. Finally, the reciprocal image $L^*\eta^*F(G^{p-1})$ is included in $F(G)$, in a natural way, as a submanifold $i: L^*\eta^*F(G^{p-1}) \to F(G)$; $(x, P^+, G|_{N_x}, b) \mapsto (x, b)$. As a fibration over $\mathscr{G}_0^p(G)$, $L^*\eta^*F(G^{p-1})$ reduces the group $SO(p)\times O(m-p)$ of $F(G)$ to the subgroup $S\Gamma(p-1, p)$. The commutative diagram below summarizes this information:

$$\begin{tikzcd}
F(G) \arrow[d, "\pi_0"] & L^*\eta^*F(G^{p-1}) \arrow[l, "i"'] \arrow[r, "L_*"] \arrow[d, "\pi_0"] & \eta^*F(G^{p-1}) \arrow[d, "\pi_0"] \\
\mathscr{G}_0^p(G) \arrow[r, leftrightarrow, "I"] & \mathscr{G}_0^p(G) \arrow[r, "L"] \arrow[dr, "d"] \arrow[ddr, "\bar{G}_*"'] & E\mathscr{G}_0^p(G^{p-1}) \arrow[d, "\eta"] \arrow[dd, bend left, "s"] \\
 & & \mathscr{G}_0^p(G^{p-1}) \arrow[d, "g_* \circ i"] \\
 & & \mathscr{G}_0^p(M)
\end{tikzcd} \tag{5.2.3}$$

Let $A_0: E\mathscr{G}_0^p(G^{p-1}) \to E\mathscr{G}_0^p(G^{p-1})$ be the antipodal map (involution); $(x, P^+, G_+) \mapsto (x, P^-, G_+)$. Then $(\mathscr{G}_0^p(G), L)$ is an A_0-invariant submanifold of $E\mathscr{G}_0^p(G^{p-1})$ that intersects each fiber $(k)^{-1}(x)$, $x \in N^p$, in exactly two points. Conversely, if $(\bar{N}^p$, inclusion) is a p-dimensional, A_0-invariant submanifold of $E\mathscr{G}_0^p(G^{p-1})$ that intersects each fiber $(k)^{-1}(x)$, $x \in N^p$, in at most two points, then $k(\bar{N}^p)$ is an open submanifold of N^p. Furthermore the formula

$$G(x) = G_x \tag{5.2.4}$$

where $x \in k(\bar{N}^p)$ and $(k)^{-1}(x) = \{(x, P^+, G_x), (x, P^-, G_x)\}$ defines a TBI G that extends (G^{p-1}, D) along $g|_{k(\bar{N}^p)}$ and $\bar{N}^p$ is the image of $\mathscr{G}_0^p(G)$ under the lift L.

PROPOSITION 5.2.1. *Let* (G^{p-1}, D) *be partial data along* $g: N^p \to M^m$.

(a) *Suppose that* G *is a parallel TBI along* g *that extends the partial data*; $G|_D = G^{p-1}$. *Then* $(\mathscr{G}_0^p(G), L)$ *is an integral manifold of the form* $\mu(G^{p-1})$ *on* $E\mathscr{G}_0^p(G^{p-1})$.

(b) *Conversely, suppose that* $(\bar{N}^p$, *inclusion*) *is a* p-*dimensional* A_0-*invariant* *integral manifold of* $\mu(G^{p-1})$ *that intersects each fiber* $(k)^{-1}(x)$, $x \in N^p$ *in at most two points. Then the TBI G defined by* (5.2.4) *is a parallel extension of the partial data* (G^{p-1}, D) *along* $g|_{k(\bar{N}^p)}$ *and* $\bar{N}^p$ *is the image of the orientation covering of* $k(\bar{N}^p) \subset N^p$ *by G under the lift L.*

Proof. (a) For any extension G of the partial data (G^{p-1}, D) it follows that $\mu(G) = \delta L \mu(G^{p-1})$. In fact, it suffices to show that $G'' = \delta L(\tau^{\#})$. But, by (3.1.1) and (5.2.3),

$$\begin{aligned} \delta L_* \tau|_{(x, G(N_x)^+, G|_{N_x}, b)}(u) &= (b|_{R^p})^{-1} \circ G|_{N_x} T(\pi_2 \circ \eta_* \circ L_*)(u) \\ &= (b|_{R^p})^{-1} \circ G|_{N_x} T\pi_1(u) = \hat{G}|_{(x,b)}(u), \end{aligned}$$

so that $\delta L_* \tau = \hat{G}|_{L \cdot \eta \cdot F(G^{p-1})}$. Thus by (1.2.5) and §3.1,

$$G'' = (\hat{G}|_{L \cdot \eta \cdot F(G^{p-1})})^{\#} = (\delta L_* \tau)^{\#} = \delta L(\tau^{\#}).$$

Hence $\mu(G) = \delta L \mu(G^{p-1})$ and part (a) follows from Proposition 3.3.1.

(b) This follows from the discussion preceding the proposition, the fact (proved in part (a)) that $\mu(G) = \delta L \mu(G^{p-1})$ and Proposition 3.3.1. Q.E.D.

5.3 *The regular integral planes of* $\mu(G^{p-1})$. Diagram (5.1.1) and equation (5.1.2) determine the following commutative diagram

$$\begin{tikzcd} E\mathscr{G}_0^p(G^{p-1}) \arrow[d, "\eta"] \arrow[dr, "s"] \arrow[dd, bend right=60, "k"'] & \\ \mathscr{G}_0^p(G^{p-1}) \arrow[r, "g_* \circ i"] \arrow[d, "p_0"] & \mathscr{G}_0^p(M) \arrow[d, "p_0"] \\ N \arrow[r, "g"] & M \end{tikzcd}$$

By Proposition 2.3.3, for each $(x, P^+) \in \mathscr{G}_0^p(G^{p-1})$, $\delta(g_* \circ i)\Phi_0$ sends the vertical subspace $V^{p_0, \mathrm{im}\, G^{p-1}}(P^+)$ isomorphically onto

$$\mathrm{Hom}\,((g_*)^* \mathscr{G}_0^{T, p-1};\ (g_* \circ i)^* \mathscr{G}_0^T, (g_* \circ i)^* \mathscr{G}_0^{\perp})(P^+) = \mathrm{Hom}\,(G^{p-1}(D(x)); P, \perp P).$$

Since η is a local diffeomorphism, the formula

$$V^k(x, P^+, G_+) = (T\eta)^{-1}(V^{p_0, \mathrm{im}\, G^{p-1}}(P^+))$$

defines a smooth $m-p$ $[=\dim \mathscr{G}_0^p(R^m, R^{p-1})]$ dimensional distribution V^k on $E\mathscr{G}_0^p(G^{p-1})$. V^k will be called the *vertical* distribution since $V^k = \mathrm{Ker}\, Tk$. Thus for each $(x, P^+, G_+) \in E\mathscr{G}_0^p(G^{p-1})$, $\delta s \Phi_0$ sends the vertical subspace $V^k(x, P^+, G_+)$ isomorphically onto

$$\mathrm{Hom}\,(\eta^*[(g_*)^* \mathscr{G}_0^{T, p-1}];\ s^* \mathscr{G}_0^T, s^* \mathscr{G}_0^{\perp})(x, P^+, G_+) = \mathrm{Hom}\,(G^{p-1}(D(x)); P, \perp P).$$

In particular, $\bar{K} \equiv \mathrm{Ker}\, \delta s \Phi_0$ is a smooth p-dimensional distribution and it is complementary to V^k. $\bar{K}$ will be called the *horizontal* distribution of $E\mathscr{G}_0^p(G^{p-1})$.

LEMMA 5.3.1. *Suppose that* $y=(x, P^+, G_+)\in E\mathcal{G}_0^p(G^{p-1})$ *and that* $E^{p-1}\subset E\mathcal{G}_0^p(G^{p-1})_y$ *is a* $p-1$ *plane complementary to the vertical and*

$$(*)\qquad TkE^{p-1}\cap \perp D(x) = \{0\}.$$

Let $v_1,\dots,v_{p-1}$ *be a basis of* E^{p-1} *and* $v_p\in \bar{K}(y)$ *such that* $Tkv_1,\dots,Tkv_p$ *is an orthonormal basis of* N_x *and*

$$(**)\qquad G_+Tk(v_1)\wedge\cdots\wedge G_+Tk(v_p) = + \quad (= \tau^\#(v_1)\wedge\cdots\wedge\tau^\#(v_p),\ by\ (5.2.2)).$$

Let $\gamma=\mathrm{Span}\,\{v_p\}$, *so that there is a direct sum decomposition*

$$E\mathcal{G}_0^p(G^{p-1})_y = E^{p-1}\oplus\gamma\oplus V^k(y).$$

Then

$$H(E^{p-1},\mu(G^{p-1})) = E^{p-1}\oplus \mathrm{Graph}\,L_\alpha\subset E\mathcal{G}_0^p(G^{p-1})_y$$

where $L_\alpha\colon \gamma\to V^k(y)$ *is the unique linear transformation such that*

$$\delta s\Phi_0\circ L_\alpha(v_p)(\tau^\#(v_p)) = -\sum_{i=1}^{p-1}\delta s\Phi_0(v_i)(\tau^\#(v_i)).$$

In particular, $H(E^{p-1},\mu(G^{p-1}))$ *is a p-dimensional complement to the vertical and* E^{p-1} *is a regular integral plane of* $\mu(G^{p-1})$.

Proof. Each $u\in E\mathcal{G}_0^p(G^{p-1})_y$ can be uniquely expressed as $u=e+cv_p+a$ where $e\in E^{p-1}$, c is real and $a\in V^k(y)$.

$$u\in H(E^{p-1},\mu(G^{p-1})) = \mathrm{Ker}\, i(v_1,\dots,v_{p-1})\mu(G^{p-1})$$

if and only if

$$\begin{aligned} 0 &= \mu(G)(v_1,\dots,v_{p-1},e+cv_p+a)\\ &= \delta s\Phi_0\wedge_{s^*s}(s^*h\circ\Lambda^{p-1}\tau^\#)(v_1,\dots,v_{p-1},cv_p+a)\quad (\text{by } (**))\\ &= (-1)^{p+1}(p-1)!\left[c\sum_{i=1}^{p-1}\delta s\Phi_0(v_i)[\tau^\#(v_i)]+\delta s\Phi_0(a)[\tau^\#(v_p)]\right]\\ &= (-1)^{p+1}(p-1)!\,[\delta s\Phi_0(a-L_\alpha(cv_p))[\tau^\#(v_p)]]. \end{aligned}$$

Equivalently,

$$\tau^\#(v_p)\in\mathrm{Ker}\,[\delta s\Phi_0(a-L_\alpha(cv_p))].$$

But (*) is equivalent to $Tk(v_p)\notin D(x)$ or $\tau^\#(v_p)=G_+Tk(v_p)\notin G_+(D(x))=G^{p-1}(D(x))$ or $\tau^\#(v_p)\in\mathrm{Ker}\,[\delta s\Phi_0(v)]$ if and only if $v=0$. Hence $u=e+cv_p+a\in H(E^{p-1},\mu(G^{p-1}))$ if and only if $a=L_\alpha(cv_p)$.

It remains to prove the regularity of E^{p-1}. Since $\mu(G^{p-1})$ is a p form every E^q, $q\leqq p-2$, is a regular integral plane. Thus it is sufficient to prove that

$$\dim H(\bar{E}^{p-1},\mu(G^{p-1}))$$

is constant in a neighborhood of E^{p-1}. But a neighborhood of E^{p-1} may be taken to be the set of $p-1$ planes that are complementary to the vertical, V^k, and whose projections intersect the distribution, $\perp D$, only in the zero vector at the foot point. But on such a neighborhood, as has been shown, $H(\bar{E}^{p-1}, \mu(G^{p-1}))$ is constantly p-dimensional. Q.E.D.

5.4 *Theorem* B (p). Let $A_0: E\mathscr{G}_0^p(G^{p-1}) \to E\mathscr{G}_0^p(G^{p-1}); (x, P^+, G_+) \mapsto (x, P^-, G_+)$ be the antipodal map defined in §5.2.

LEMMA 5.4.1. *If $E^p \subset E\mathscr{G}_0^p(G^{p-1})_{(x,P^+,G_+)}$ is an integral p plane of $\mu(G^{p-1})$ and is complementary to the vertical, then the same is true for*

$$TA_0(E^p) \subset E\mathscr{G}_0^p(G^{p-1})_{(x,P^-,G_+)}.$$

Proof. $\eta^*F(G^{p-1})$ has the structure of a principal $O(p-1)\times O(m-p+1)\times O(1)$ bundle over N with projection $\pi_2 \circ \eta_*$. In fact, let $\varphi_{\mathscr{U}}: O(p-1)\times O(m-p+1) \times \mathscr{U} \to \pi_2^{-1}(\mathscr{U})$ be a strip map for $F(G^{p-1})$ over $\mathscr{U}\subset N$ and let Y be a unit vector field on $\mathscr{U}$ that is orthogonal to D. Then define a strip map for $\eta^*F(G^{p-1})$ over $\mathscr{U}$ by

$$\tilde{\varphi}_{\mathscr{U}}: O(p-1)\times O(m-p+1)\times O(1)\times \mathscr{U} \to (\pi_2\circ\eta_*)^{-1}\mathscr{U}$$

by

$$\tilde{\varphi}_{\mathscr{U}}(\sigma, \pm 1, x) \to (\varphi_{\mathscr{U}}(\sigma, x), G_{\pm})$$

where G_+ and G_- are the extensions of $G^{p-1}|_{D(x)}$ to isometries $N_x \to \varphi_{\mathscr{U}}(\sigma, x)(R^p)$ such that $G_{\pm}(Y(x)) = \pm e_p(\sigma, x) = \pm$ [the pth vector in the frame $\varphi_{\mathscr{U}}(\sigma, x)$]. Thus, $(x, b, G_+)(\sigma, \pm 1) = (x, b\sigma, G_{\pm})$ and the right action of $S\Gamma(p-1, p)$ on $\eta^*F(G^{p-1})$ as a principal bundle over $E\mathscr{G}_0^p(G^{p-1})$ is obtained by restricting the action of $O(p-1)\times O(m-p+1)\times O(1)$ to the subgroup $S\Gamma(p-1,p)\times\{1\}$. Let

$$\Delta: O(p-1)\times O(m-p+1)\times O(1) \to O(p-1)\times O(m-p)$$

be the projection map.

As in the proof of Lemma 4.3.1 fix $\sigma \in \Gamma(p-1,p) - S\Gamma(p-1,p)$ where $\sigma_{ij} = (-1)^{\delta(i,p)}\cdot\delta(i,j)$. If $(x, b, G_+) \in \eta^*F(G^{p-1})$ then $b(R^p) = b\sigma(R^p)$ but

$$\Lambda^p(b)(r_1 \wedge \cdots \wedge r_p) = -\Lambda^p(b\sigma)(r_1 \wedge \cdots \wedge r_p),$$

hence

$$\text{(I)} \qquad \pi_0 \circ r_{(\sigma,1)} = A_0 \circ \pi_0: \eta^*F(G^{p-1}) \to E\mathscr{G}_0^p(G^{p-1}).$$

Referring to (5.1.2) and letting

$$\bar{s} = g_* \circ i_* \circ \eta_*: \eta^*F(G^{p-1}) \to F(M),$$

it follows that $\bar{s}\circ r_{(\sigma,1)} = r_\sigma \circ \bar{s}$. Consequently by the Hom (ρ_p, ρ_{m-p}) equivariance of $\varphi_{p,m-p}$:

$$\text{(II)} \qquad \delta r_{(\sigma,1)}\delta\bar{s}\varphi_{p,m-p} = \operatorname{Hom}(\rho_p, \rho_{m-p}) \circ \Delta(\sigma^{-1}, 1)\delta\bar{s}\varphi_{p,m-p}.$$

Furthermore, in the same manner as the equivariance of τ was proved in §5.2, it can be seen that

$$\text{(III)} \qquad \delta r_{(\sigma,1)}\tau = \rho_p \circ \Delta(\sigma^{-1}, 1)\tau.$$

The lemma now follows from (I), (II), and (III) by using the same technique as in the proof of the A_0-invariance of the integral p-planes of μ in Lemma 4.3.1. Q.E.D.

Let (G^{p-1}, D) be partial data along $g\colon N^p \to M$. Let $(N^{p-1}, i = \text{inclusion})$ be a submanifold of N^p and $G^p\colon i^*T(N^p) \to T(M)$ a Euclidean map along $g \circ i$. G^{p-1} induces a map $G^{p-1} \circ i_*\colon i^*D \to T(M)$ along $g \circ i$. (G^{p-1}, D) and G^p are *compatible* if $G^p|_{i^*D} = G^{p-1} \circ i_*$.

THEOREM B (p) 5.4.2. *Let $g\colon N^p \to M^m$ be an (not necessarily isometric) immersion of Riemannian manifolds. Let D be a $(p-1)$ dimensional distribution on N^p and $(N^{p-1}, i = \text{inclusion})$ a homeomorphically embedded submanifold of N^p transversal to the* 1-*dimensional distribution $\perp D$. Suppose that*

$G^{p-1}\colon D \to T(M)$ is a Euclidean map along g,

$G^p\colon i^*T(N^p) \to T(M)$ is a Euclidean map along $g \circ i$ and (G^{p-1}, D) and G^p are compatible.

Then, assuming that all the data is real analytic, there is a neighborhood $\mathscr{U}$ of N^{p-1} in N^p and a unique real analytic parallel TBI $G\colon T(\mathscr{U}) \to T(M)$ that extends the initial data

$$G|_D = G^{p-1}\colon D \to T(M) \text{ along } g|_{\mathscr{U}}$$

and

$$G \circ i_* = G^p\colon i^*T(N^p) \to T(M) \text{ along } g \circ i.$$

Proof. G^p determines a lift $\bar{G}^p\colon N^{p-1} \to \mathscr{G}^p(G^{p-1})$ (see §5.1 for the definitions); $x \mapsto (x, G^p[i^*T(N^p)(x)])$. $\bar{N}^{p-1} \equiv (\bar{G}^p)^*\mathscr{G}_0^p(G^{p-1})$ is a two fold cover of N^{p-1}. Since G^{p-1} and G^p are compatible, there is a lift $l\colon \bar{N}^{p-1} \to E\mathscr{G}_0^p(G^{p-1})$;

$$(x, G^p[i^*T(N^p)(x)]^+) \mapsto (x, G^p[i^*T(N^p)(x)]^+, G^p \circ i_*^{-1}|_{N_x}).$$

Thus the following diagram commutes:

$$\begin{tikzcd}
 & E\mathscr{G}_0^p(G^{p-1}) \arrow[d, "\eta"] \\
 & \mathscr{G}_0^p(G^{p-1}) \arrow[d, "z"] \\
(G^p)^*\mathscr{G}_0^p(G^{p-1}) = \bar{N}^{p-1} \arrow[uur, "l"] \arrow[ur, "\bar{G}^p_*"] \arrow[d, "z"] & \mathscr{G}^p(G^{p-1}) \arrow[d, "p_1"] \\
N^{p-1} \arrow[ur, "\bar{G}^p"] \arrow[r, "i"] & N^p
\end{tikzcd}$$

The proof of the local part of the theorem will proceed in the same way as the proof of Theorem A (p) 4.3.2. Let $x_0 \in N^{p-1} \subset N^p$ be given and choose

$$y_+ = (x_0, G^p[i^*T(N^p)(x_0)]^+) \in \bar{N}^{p-1}(x_0).$$

$E_+^{p-1} \equiv Tl(\bar{N}_{y_+}^{p-1})$ is a $p-1$ plane complementary to the vertical. $Tk(E_+^{p-1}) \cap \perp D(x_0) = \{0\}$ because of the transversality assumption and the commutativity of the diagram above. By Lemma 5.3.1 E_+^{p-1} is a regular integral plane of $\mu(G^{p-1})$ and $H(E_+^{p-1}, \mu(G^{p-1}))$ is a p-dimensional complement to the vertical $V^k(l(y_+))$. By the Cartan-Kahler Theorem 1.4.1, there is a neighborhood $\mathscr{U}_+$ of $l(y_+)$ in $E\mathscr{G}_0^p(G^{p-1})$ and a unique real analytic p-dimensional integral manifold $\bar{N}_+^p$ of $\mu(G^{p-1})$ passing through $l(y_+)$, contained in $\mathscr{U}_+$ and containing the connected component of $l(\bar{N}^{p-1}) \cap \mathscr{U}_+$ through $l(y_+)$. Assume that $\mathscr{U}_+$ is sufficiently small so that $\bar{N}_+^p$ projects diffeomorphically under k onto its image, $\mathscr{U}^{x_0}$. In particular $\bar{N}_+^p \cap A_0\bar{N}_+^p$ is empty. Let $y_- = (x_0, G^p[i^*T(N^p)(x_0)]^+)$ be the other point of $\bar{N}^{p-1}(x_0)$; $A_0 l(y_+) = l(y_-)$. Let $E_-^{p-1} = TA_0E_+^{p-1}$, $\mathscr{U}_- = A_0\mathscr{U}_+$ and $\bar{N}_-^p = A_0\bar{N}_+^p$. By Lemma 5.4.1, $\bar{N}_-^p$ is an integral manifold of $\mu(G^{p-1})$. By the uniqueness in the Cartan-Kahler Theorem, $\bar{N}_-^p$ is the unique p-dimensional integral manifold of $\mu(G^{p-1})$ passing through $l(y_-)$, contained in $\mathscr{U}_-$ and containing the connected component of $l(\bar{N}_-^p) \cap \mathscr{U}_-$ passing through $l(y_-)$. Let $\bar{N}^p = \bar{N}_+^p \cup \bar{N}_-^p$. By Proposition 5.2.1(b) the tangent bundle isometry $G \equiv G^{x_0}$ defined by equation (5.2.4) is a parallel extension of the partial data G^{p-1} along $g|_{\mathscr{U}^{x_0}}$. Furthermore $\bar{N}^p$ is the image of the orientation covering of $\mathscr{U}^{x_0} \subset N^p$ by G^{x_0} under the lift L (see diagram (5.2.3)). In particular, G^{x_0} extends the initial data G^p along the connected component of $\mathscr{U}^{x_0} \cap N^{p-1}$ through x_0. The uniqueness of the solution G^{x_0} on $\mathscr{U}^{x_0}$ follows again from the Cartan-Kahler Theorem.

Using this local uniqueness of a parallel extension of the initial data together with the assumption that N^{p-1} is homeomorphically embedded, it is a routine matter to construct a neighborhood $\mathscr{U}$ of N^{p-1} in N^p with $\mathscr{U} \subset \bigcup_{x_0 \in N^{p-1}} \mathscr{U}^{x_0}$ and a unique parallel TBI G along $g|_{\mathscr{U}}$ extending the initial data such that $G|_{\mathscr{U} \cap \mathscr{U}^{x_0}} = G^{x_0}|_{\mathscr{U} \cap \mathscr{U}^{x_0}}$ for each $x_0 \in N^{p-1}$. Q.E.D.

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